TILINGS AND ASSOCIATED RELATIONAL STRUCTURES

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Abstract. In the present paper, as we did previously in [7], we investigate the relations between the geometric properties of tilings and the algebraic properties of associated relational structures. Our study is motivated by the existence of aperiodic tiling systems. In [7], we considered tilings of the euclidean spaces $\mathbb{R}^k$, and isomorphism was defined up to translation. Here, we consider, more generally, tilings of a metric space, and isomorphism is defined modulo an arbitrary group of isometries.

In Section 1, we define the relational structures associated to tilings. The results of Section 2 concern local isomorphism, the extraction preorder and the characterization of relational structures which can be represented by tilings of some given type.

In Section 3, we show that the notions of periodicity and invariance through a translation, defined for tilings of the euclidean spaces $\mathbb{R}^k$, can be generalized, with appropriate hypotheses, to relational structures, and in particular to tilings of noneuclidean spaces.

The results of Section 4 are valid for uniformly locally finite relational structures, and in particular tilings, which satisfy the local isomorphism property. We characterize among such structures those which are locally isomorphic to structures without nontrivial automorphism. We show that, in an euclidean space $\mathbb{R}^k$, this property is true for a tiling which satisfies the local isomorphism property if and only if it is not invariant through a nontrivial translation. We illustrate these results with examples, some of them concerning aperiodic tiling systems of euclidean or noneuclidean spaces.

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In Section 1, we consider finite systems consisting of prototiles in a metric space E and local rules for assembling tiles which are equivalent to the prototiles modulo a group G of isometries of E. With appropriate hypotheses, we associate to each such system Δ a finite relational language $\mathcal{L}_\Delta$ such that any Δ-tiling can be interpreted as a $\mathcal{L}_\Delta$-structure.

In [7], we did the same thing in a particular case: E was an euclidean space $\mathbb{R}^k$ and G consisted of the translations of E. In that case, $\mathcal{L}_\Delta$ can be defined by considering all the possible configurations of two adjacent tiles. In

the present situation, we have to consider the possible configurations for the patch consisting of all tiles within some given "distance" of one of them; that "distance" is depending on E, G and Δ.

In the following sections, we prove various properties of Δ-tilings, some of them classical and others new, concerning local isomorphism, invariance through a translation... Classically, such results are shown by considering the geometrical and topological properties of the space and the tilings. The proofs given here are obtained by considering the algebraic properties of the associated $\mathcal{L}_\Delta$-structures. In that way, we prove the results for larger classes of tilings, and we show similar properties for relational structures which are not represented by tilings (see Section 4, Example 4).

It is also natural to wonder, for a given system Δ, which $\mathcal{L}_\Delta$-structures can be represented by Δ-tilings. Characterizations of such structures are given in Section 2.

1. Tilings and associated relational structures.

In [7], we considered relational structures associated to tilings of the euclidean spaces $\mathbb{R}^k$. The isomorphism of tiles, patches, tilings... was defined up to translation.

Actually, much more various situations have been investigated. For instance, the isomorphism of tiles, patches, tilings... can be defined modulo a group of isometries of $\mathbb{R}^k$ which contains symmetries and/or rotations. Also, tilings of noneuclidean spaces and tilings with partially overlapping tiles are considered.

The following facts appear to be common to all cases:

- any tiling is a covering of a metric space (E, δ) by tiles which are obtained from a finite set of closed bounded prototiles by applying isometries belonging to some specified group;
- up to isometries in the group, only finitely many configurations of some given size can appear in a tiling;
- the work of a tiler can be described as follows: first he puts a small cluster of tiles somewhere in the space, then he gradually increases the patch by adding new tiles; at each step, he applies the same finite set of local rules, which only leave finitely many possibilities for adding new tiles.

In order to formalize these facts, we introduce some definitions and notations. We consider a metric space (E, δ) and a group G of bijective isometries of E. We do not suppose that (E, δ) is homogeneous, or that local isometries of E can be extended to global ones.

We call a *tile* any closed bounded subset T of E. We do not suppose T connected or T equal to the closure of its interior. For any tiles T, T' (resp. any sets of tiles $\mathcal{E}, \mathcal{E}'$), we say that T and T' (resp. $\mathcal{E}$ and $\mathcal{E}'$) are *isomorphic* if there exists $\sigma \in G$ such that $T\sigma = T'$ (resp. $\mathcal{E}\sigma = \mathcal{E}'$). We

define isomorphism in the same way for the pairs (S,T) and the pairs $(\mathcal{E},T)$ with S,T tiles and $\mathcal{E}$ a set of tiles.

Remark. Sometimes, tiles with drawings are also considered. A *drawing* is a map from a tile T to a finite set Ω, whose elements are called *colours*. In that case, the homomorphisms that we consider must respect colours; moreover, in the definitions of configuration, tiling and patch given below, any point in the intersection of two tiles must have the same colour in each of them. The results of the present paper are proved in the same way for tiles with drawings.

For each set $\mathcal{E}$ of tiles and each $T \in \mathcal{E}$, we define inductively the subsets $\mathcal{B}_n^{\mathcal{E}}(T)$ with $\mathcal{B}_0^{\mathcal{E}}(T) = \{T\}$ and, for each $n \in \mathbb{N}$,
$\mathcal{B}_{n+1}^{\mathcal{E}}(T) = \{U \in \mathcal{E} \mid \text{there exists } V \in \mathcal{B}_n^{\mathcal{E}}(T) \text{ such that } U \cap V \neq \emptyset\}$.
For any tiles T,T' and any sets of tiles $\mathcal{E},\mathcal{E}'$, we write $(\mathcal{E},T) \leq (\mathcal{E}',T')$ if there exists $\sigma \in G$ such that $\mathcal{E}\sigma \subset \mathcal{E}'$ and $T\sigma = T'$.

We call a *tiling* any covering $\mathcal{E}$ of E by possibly overlapping tiles such that, for each $r \in \mathbb{N}^*$:
1) for each $T \in \mathcal{E}$, $\mathcal{B}_r^{\mathcal{E}}(T)$ is finite and $\mathcal{B}_{r-1}^{\mathcal{E}}(T)$ is contained in the interior of the union of the tiles of $\mathcal{B}_r^{\mathcal{E}}(T)$;
2) for any $S,T \in \mathcal{E}$, if $(\mathcal{B}_r^{\mathcal{E}}(S),S) \leq (\mathcal{B}_r^{\mathcal{E}}(T),T)$, then $(\mathcal{B}_r^{\mathcal{E}}(S),S) \cong (\mathcal{B}_r^{\mathcal{E}}(T),T)$;
3) the pairs $(\mathcal{B}_r^{\mathcal{E}}(T),T)$ for $T \in \mathcal{E}$ fall in finitely many isomorphism classes.

If 1) is true for $r=1$, then it is true for each $r \in \mathbb{N}^*$. If 1) is true and if 2) is true for $r=1$, then, for any $S,T \in \mathcal{E}$, each $\sigma \in G$ such that $\mathcal{B}_1^{\mathcal{E}}(S)\sigma \subset \mathcal{B}_1^{\mathcal{E}}(T)$ and $S\sigma = T$ is an isomorphism from $(\mathcal{B}_1^{\mathcal{E}}(S),S)$ to $(\mathcal{B}_1^{\mathcal{E}}(T),T)$. By induction, it follows that, for each $r \in \mathbb{N}^*$ and any $S,T \in \mathcal{E}$, each $\sigma \in G$ such that $\mathcal{B}_r^{\mathcal{E}}(S)\sigma \subset \mathcal{B}_r^{\mathcal{E}}(T)$ and $S\sigma = T$ is an isomorphism from $(\mathcal{B}_r^{\mathcal{E}}(S),S)$ to $(\mathcal{B}_r^{\mathcal{E}}(T),T)$. Consequently, 2) is true for each $r \in \mathbb{N}^*$. We note that 2) is necessarily true if 1) is true and if the tiles of $\mathcal{E}$ are equal to the closure of their interior and nonoverlapping.

It is natural to wonder whether 3) is also true for each $r \in \mathbb{N}^*$ if it is true for $r=1$. If (E,δ) is an euclidean space $\mathbb{R}^k$ and if G consists of the translations of $\mathbb{R}^k$, then any pair $(S,T) \in \mathcal{E} \times \mathcal{E}$ is completely determined by S and the isomorphism class of (S,T). Consequently, if the pairs $(S,T) \in \mathcal{E} \times \mathcal{E}$ with $S \cap T \neq \emptyset$ fall in finitely many isomorphism classes, then, for each $n \in \mathbb{N}^*$, the pairs $(\mathcal{B}_n^{\mathcal{E}}(T),T)$ for $T \in \mathcal{E}$ fall in finitely many isomorphism classes. The following example shows that the situation is different if G consists of the isometries of $\mathbb{R}^k$:

Example 1. Let (E,δ) be the euclidean space $\mathbb{R}^2$ and let G consist of the isometries of (E,δ). Consider the coverings of $\mathbb{R}^2$ by nonoverlapping tiles isomorphic to the following prototiles:
$T_0 = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } x^2+y^2 \leq 1\}$,

$T_k = \{(x,y) \in \mathbb{R}^2 \mid k^2 \leq x^2 + y^2 \leq (k+1)^2\}$ for $1 \leq k \leq 2n+1$,
$T_{2n+2} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq (2n+2)^2$ and $\sup(|x|, |y|) \leq 2n+3\}$, with any two copies of T_{2n+2} having one edge in common if they have more than one common point.

Each such covering consists of squares with sides of length $4n+6$, each of them covered by two copies of T_0 and one copy of each of the tiles $T_1, ..., T_{2n+2}$. Moreover, in each square, the two copies of T_0 can be rotated together arbitrarily. Consequently, in such a covering $\mathcal{E}$, the pairs $(\mathcal{B}^{\mathcal{E}}_{n+1}(T), T)$ for $T \in \mathcal{E}$ do not generally fall in finitely many isomorphism classes, contrary to the pairs $(\mathcal{B}^{\mathcal{E}}_n(T), T)$.

Now, we can ask the following question: Suppose that we specify a finite set of isomorphism classes for the pairs $(\mathcal{B}^{\mathcal{E}}_1(T), T)$ which can appear in a covering $\mathcal{E}$ of E which satisfies 1) and 2). Is there an integer m such that, if the pairs $(\mathcal{B}^{\mathcal{E}}_m(T), T)$ for $T \in \mathcal{E}$ fall in finitely many isomorphism classes, then, for each $n \in \mathbb{N}^*$, the pairs $(\mathcal{B}^{\mathcal{E}}_n(T), T)$ for $T \in \mathcal{E}$ fall in finitely many isomorphism classes.

Example 2 below shows that this property does not necessarily hold. On the other hand, we are going to prove (see Proposition 1.10) that some natural conditions on (E, δ) and G, satisfied by the euclidean spaces $\mathbb{R}^k$, and on the pairs $(\mathcal{B}^{\mathcal{E}}_1(T), T)$ for $T \in \mathcal{E}$, are enough to make it true.

Example 2. Let $E = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ be the surface of a cylinder of infinite length. Define the distance δ by considering E as a quotient of the euclidean space $\mathbb{R}^2$. Let G consist of the isometries of E. Write $T_1 = \{(x,y,z) \in E \mid x \geq 0$ and $0 \leq z \leq 1\}$ and $T_2 = \{(x,y,z) \in E \mid 0 \leq z \leq 1\}$. For each $n \in \mathbb{N}^*$, consider the coverings $\mathcal{E}$ of E by nonoverlapping tiles obtained as follows: for each $a \in \mathbb{Z}$, $\{(x,y,z) \in E \mid a \leq z \leq a+1\}$ is covered by two tiles isomorphic to T_1 if $a \in (2n+2)\mathbb{Z}$ and by one tile isomorphic to T_2 otherwise. For each $a \in (2n+2)\mathbb{Z}$, the two tiles covering $\{(x,y,z) \in E \mid a \leq z \leq a+1\}$ can be rotated together arbitrarily. Consequently, in such a covering $\mathcal{E}$, the pairs $(\mathcal{B}^{\mathcal{E}}_{n+1}(T), T)$ for $T \in \mathcal{E}$ do not generally fall in finitely many isomorphism classes, contrary to the pairs $(\mathcal{B}^{\mathcal{E}}_n(T), T)$. Moreover, the pairs $(\mathcal{B}^{\mathcal{E}}_1(T), T)$ for all possible choices of $\mathcal{E}$ and T fall in three isomorphism classes.

Now we state the definitions, the notations and the hypotheses which are necessary for our results. For each $x \in E$ and each $\eta \in \mathbb{R}_{\geq 0}$, we write $\beta(x, \eta) = \{y \in E \mid \delta(x,y) \leq \eta\}$. For each nonempty bounded $S \subset E$, we consider the *radius* $\mathrm{Rad}(S) = \inf_{x \in E}(\sup_{y \in S} \delta(x,y))$ and the *diameter* $\mathrm{Diam}(S) = \sup_{x,y \in S} \delta(x,y)$; we have $\mathrm{Diam}(S) \leq 2.\mathrm{Rad}(S)$.

From now on, we suppose that E is connected and that $\beta(x, \eta)$ is compact for each $x \in E$ and each $\eta \in \mathbb{R}_{>0}$. Then each bounded closed subset (i.e. tile)

of E is compact.

For each $x \in E$ and each $\xi \in \mathbb{R}_{>0}$, we define inductively $\beta(x,\xi,n)$ with $\beta(x,\xi,0) = \{x\}$ and, for each $n \in \mathbb{N}$, $\beta(x,\xi,n+1) = \cup_{y\in\beta(x,\xi,n)}\beta(y,\xi)$. We have $\cup_{n\in\mathbb{N}}\beta(x,\xi,n) = E$ since E is connected and $\cup_{n\in\mathbb{N}}\beta(x,\xi,n)$ is both open and closed. For each $n \in \mathbb{N}$, we write $\omega(x,\xi,n) = \sup\{\eta \in \mathbb{R}_{\geq 0} \mid \beta(x,\eta) \subset \beta(x,\xi,n)\}$.

Now we show that $\omega(x,\xi,n)$ tends to infinity with n. Otherwise, there exists $\eta \in \mathbb{R}_{>0}$ such that $\beta(x,\eta) \nsubseteq \beta(x,\xi,n)$ for each $n \in \mathbb{N}$. It follows that $\beta(x,\eta)$ contains a sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \notin \beta(x,\xi,n)$ for each $n \in \mathbb{N}$. As $\beta(x,\eta)$ is compact, there exists a subsequence $(z_n)_{n\in\mathbb{N}}$ which converges to a point $z \in \beta(x,\eta)$. For n large enough, we have $\delta(z,z_n) < \xi$, which implies that $z \notin \beta(x,\xi,n-1)$ since $z_n \notin \beta(x,\xi,n)$. It follows that $z \notin \cup_{n\in\mathbb{N}}\beta(x,\xi,n)$, contrary to what we proved just above.

We say that (E,δ) is:
- *weakly homogeneous* if $\omega(\xi,n) = \inf_{x\in E}\omega(x,\xi,n)$ tends to infinity with n;
- *geodesic* if any two points of E can be joined by at least one geodesic;
- *transitive* if, for any $x,y \in E$, there exists an isometry σ such that $x\sigma = y$.

Any geodesic space is weakly homogeneous since $\omega(x,\xi,n) = n\xi$ for any x,ξ,n. Any transitive space is also weakly homogeneous since $\omega(x,\xi,n)$ does not depend on x. In some of our results, we shall suppose that (E,δ) is weakly homogeneous or geodesic.

For each *finite* set $\mathcal{E}$ of tiles, each $T \in \mathcal{E}$ and each $p \in \mathbb{N}^*$, we say that $(\mathcal{E},T)$ is a *p-configuration* if $\mathcal{B}^{\mathcal{E}}_p(T) = \mathcal{E}$ and if each $S \in \mathcal{B}^{\mathcal{E}}_{p-1}(T)$ is contained in the interior of the union of the tiles of $\mathcal{B}^{\mathcal{E}}_1(S)$. Then $\mathcal{B}^{\mathcal{E}}_{p-1}(T)$ is contained in the interior of the union of the tiles of $\mathcal{E}$ and $(\mathcal{B}^{\mathcal{E}}_k(S),S)$ is a k-configuration for each $k \in \{1,...,p\}$ and each $S \in \mathcal{B}^{\mathcal{E}}_{p-k}(T)$.

Lemma 1.1. For each 1-configuration $(\mathcal{E},T)$, there exists $\xi \in \mathbb{R}_{>0}$ such that $\cup_{x\in T}\beta(x,\xi)$ is contained in the union of the tiles of $\mathcal{E}$.

Proof. For each $n \in \mathbb{N}^*$, consider $x_n \in T$ such that $\beta(x_n,1/n)$ is not contained in the union of the tiles of $\mathcal{E}$. As T is compact, there exists a subsequence of $(x_n)_{n\in\mathbb{N}}$ which converges to a point $x \in T$, and x does not belong to the interior of the union of the tiles of $\mathcal{E}$, whence a contradiction. ■

Proposition 1.2. Let $\mathcal{E}$ be a nonempty set of tiles such that the pairs $(\mathcal{B}^{\mathcal{E}}_1(T),T)$ for $T \in \mathcal{E}$ are 1-configurations and fall in finitely many isomorphism classes. Then
1) $\mathcal{E}$ covers E and $\{S \in \mathcal{E} \mid S\cap\beta(x,\eta) \neq \varnothing\}$ is finite for each $x \in E$ and each $\eta \in \mathbb{R}_{>0}$;
2) If (E,δ) is weakly homogeneous, then, for each $\eta \in \mathbb{R}_{>0}$, there exists $r \in \mathbb{N}^*$ such that $\delta(S,T) \leq \eta$ implies $T \in \mathcal{B}^{\mathcal{E}}_r(S)$ for any $S,T \in \mathcal{E}$.

Proof. By Lemma 1.1, there exists $\xi \in \mathbb{R}_{>0}$ such that $\mathcal{B}_1^{\mathcal{E}}(T)$ covers $\beta(x,\xi)$ for each $T \in \mathcal{E}$ and each $x \in T$. Consequently, the union of the tiles of $\mathcal{E}$ is both open and closed, and $\mathcal{E}$ covers E since E is connected.

Now suppose that there exist $x \in E$ and $\eta \in \mathbb{R}_{>0}$ such that $\{S \in \mathcal{E} \mid S \cap \beta(x,\eta) \neq \varnothing\}$ is infinite. Consider a sequence $(y_n)_{n\in\mathbb{N}} \subset \beta(x,\eta)$ and a sequence of distinct tiles $(T_n)_{n\in\mathbb{N}} \subset \mathcal{E}$ such that $y_n \in T_n$ for each $n \in \mathbb{N}$. As $\beta(x,\eta)$ is compact, there exists a subsequence of $(y_n)_{n\in\mathbb{N}}$ which converges to a point $y \in \beta(x,\eta)$. For each $T \in \mathcal{E}$ such that $y \in T$, $\mathcal{B}_2^{\mathcal{E}}(T)$ contains infinitely many T_n since $\mathcal{B}_1^{\mathcal{E}}(T)$ covers $\beta(x,\xi)$, whence a contradiction.

The second part of Proposition 1.2 is true because, for each $n \in \mathbb{N}$, each $S \in \mathcal{E}$ and each $x \in S$, $\mathcal{B}_n^{\mathcal{E}}(S)$ covers $\beta(x,\omega(\xi,n))$ and $\mathcal{B}_{n+1}^{\mathcal{E}}(S)$ contains $\{T \in \mathcal{E} \mid T \cap \beta(x,\omega(\xi,n)) \neq \varnothing\}$. ■

For each $p \in \mathbb{N}^*$, we call a *p-local rule* any set $\Gamma = \{(\mathcal{C}_1, C_1), ..., (\mathcal{C}_m, C_m)\}$ of pairwise nonisomorphic p-configurations such that:
1) for any $i, j \in \{1, ..., m\}$, if $(\mathcal{B}_1^{\mathcal{C}_i}(C_i), C_i) \leq (\mathcal{B}_1^{\mathcal{C}_j}(C_j), C_j)$, then $(\mathcal{B}_1^{\mathcal{C}_i}(C_i), C_i) \cong (\mathcal{B}_1^{\mathcal{C}_j}(C_j), C_j)$;
2) for each $i \in \{1, ..., m\}$ and each $S \in \mathcal{B}_1^{\mathcal{C}_i}(C_i)$, there exists $j \in \{1, ..., m\}$ such that $(\mathcal{B}_{p-1}^{\mathcal{C}_i}(S), S) \cong (\mathcal{B}_{p-1}^{\mathcal{C}_j}(C_j), C_j)$.
If Γ is a p-local rule, then, for each $q \in \{1, ..., p-1\}$, any representatives of the isomorphism classes of $(\mathcal{B}_q^{\mathcal{C}_1}(C_1), C_1), ..., (\mathcal{B}_q^{\mathcal{C}_m}(C_m), C_m)$ form a q-local rule.

We say that a set $\mathcal{E}$ of tiles *satisfies* Γ if, for each $T \in \mathcal{E}$, the pair $(\mathcal{B}_p^{\mathcal{E}}(T), T)$ is isomorphic to one of the pairs $(\mathcal{C}_i, C_i)$. Each tiling satisfies a p-local rule for each $p \in \mathbb{N}^*$.

We say that a set $\mathcal{E}$ of tiles is a *patch* if there exist no $\mathcal{F}, \mathcal{G} \subset \mathcal{E}$ such that $\mathcal{F} \cup \mathcal{G} = \mathcal{E}$ and $S \cap T = \varnothing$ for each $S \in \mathcal{F}$ and each $T \in \mathcal{G}$. This property is true if and only if, for any $S, T \in \mathcal{E}$, there exists $r \in \mathbb{N}$ such that $T \in \mathcal{B}_r^{\mathcal{E}}(S)$.

For each finite set $\mathcal{E}$ of tiles and each connected set $A \subset E$, if A is contained in the union of the tiles of $\mathcal{E}$, and if each tile of $\mathcal{E}$ contains a point of A, then $\mathcal{E}$ is a patch. The union of the tiles of a patch is not necessarily connected if the tiles themselves are not connected.

For each set $\mathcal{E}$ of tiles, we write $\mathrm{I}_\Gamma(\mathcal{E}) = \cup_{1\leq i\leq m}\{T \in \mathcal{E} \mid (\mathcal{B}_p^{\mathcal{E}}(T), T) \cong (\mathcal{C}_i, C_i)\}$. We say that $\mathcal{E}$ is a Γ*-patch* if:
1) for each $T \in \mathcal{E}$, there exists $i \in \{1, ..., m\}$ such that $(\mathcal{B}_p^{\mathcal{E}}(T), T) \leq (\mathcal{C}_i, C_i)$.
2) $\mathrm{I}_\Gamma(\mathcal{E})$ contains a patch $\mathcal{A}$ such that $\mathcal{E} = \cup_{T\in\mathcal{A}}\mathcal{B}_p^{\mathcal{E}}(T)$.

For each set $\mathcal{E}$ of tiles which satisfies Γ, each $T \in \mathcal{E}$ and each integer $k \geq p$, $\mathcal{B}_k^{\mathcal{E}}(T)$ is a Γ-patch since $\mathcal{B}_{k-p}^{\mathcal{E}}(T)$ is connected, $\mathcal{B}_{k-p}^{\mathcal{E}}(T) \subset \mathrm{I}_\Gamma(\mathcal{B}_k^{\mathcal{E}}(T))$ and $\mathcal{B}_k^{\mathcal{E}}(T) = \cup_{S\in\mathcal{B}_{k-p}^{\mathcal{E}}(T)}\mathcal{B}_p^{\mathcal{E}}(S)$.

Now, for each $q \in \mathbb{N}^*$ and each q-local rule $\Delta = \{(\mathcal{D}_1, D_1), ..., (\mathcal{D}_n, D_n)\}$ such that each $\mathcal{B}_{q-1}^{\mathcal{D}_i}(D_i)$ is fixed by no $\sigma \in G - \{\mathrm{Id}\}$, we define a finite

relational language $\mathcal{L}_\Delta$ such that the Δ-patches can be represented by $\mathcal{L}_\Delta$-structures. As a consequence, we show that any set of tiles which satisfies Δ is a tiling. Practically, in many examples of tilings, each 1-configuration is fixed by no $\sigma \in G - \{\mathrm{Id}\}$, so that we can take Δ with $q = 2$.

For each $i \in \{1, ..., n\}$, we write $\mathcal{D}_i = \{D_{i,1}, ..., D_{i,p(i)}\}$ with $D_{i,1} = D_i$, and we introduce a $p(i)$-ary relational symbol $R_i(u_{i,1}, ..., u_{i,p(i)})$. We write $\mathcal{L}_\Delta = \{R_1, ..., R_n\}$.

For each set $\mathcal{E}$ of tiles, we define a $\mathcal{L}_\Delta$*-structure* on $\mathcal{E}$ as follows: for $1 \leq i \leq n$ and $T_1, ..., T_{p(i)} \in \mathcal{E}$, we write $R_i(T_1, ..., T_{p(i)})$ if $\mathcal{B}_q^{\mathcal{E}}(T_1) = \{T_1, ..., T_{p(i)}\}$ and if there exists $\sigma \in G$ such that $D_{i,j}\sigma = T_j$ for $1 \leq j \leq p(i)$. For $1 \leq i, j \leq n$ and $S, S_2, ..., S_{p(i)}, T_2, ..., T_{p(j)} \in \mathcal{E}$, the relations $R_i(S, S_2, ..., S_{p(i)})$ and $R_j(S, T_2, ..., T_{p(j)})$ imply $i = j$ and $\{S_2, ..., S_{p(i)}\} = \{T_2, ..., T_{p(i)}\}$.

Any set of tiles $\mathcal{E}$ satisfies Δ if and only if, for each $T \in \mathcal{E}$, there exist $1 \leq i \leq n$ and $T_2, ..., T_{p(i)} \in \mathcal{E}$ such that $R_i(T, T_2, ..., T_{p(i)})$. For any Δ-patches $\mathcal{E} \subset \mathcal{F}$, the $\mathcal{L}_\Delta$-structure defined on $\mathcal{E}$ is the restriction to $\mathcal{E}$ of the $\mathcal{L}_\Delta$-structure defined on $\mathcal{F}$.

The following theorem implies that any Δ-patch, and in particular any set of tiles which satisfies Δ, is determined up to isomorphism by the associated $\mathcal{L}_\Delta$-structure:

Theorem 1.3. For any Δ-patches $\mathcal{E}, \mathcal{F}$ and each $\mathcal{L}_\Delta$-homomorphism $f : \mathcal{E} \to \mathcal{F}$, there exists a unique $\sigma \in G$ such that $Sf = S\sigma$ for each $S \in \mathcal{E}$.

Proof. Consider a patch $\mathcal{A} \subset \mathrm{I}_\Delta(\mathcal{E})$ such that $\mathcal{E} = \cup_{T \in \mathcal{A}} \mathcal{B}_q^{\mathcal{E}}(T)$. For each $T \in \mathcal{A}$, f induces a bijection from $\mathcal{B}_q^{\mathcal{E}}(T)$ to $\mathcal{B}_q^{\mathcal{F}}(Tf)$, and there exists a unique $\sigma_T \in G$ such that $S\sigma_T = Sf$ for each $S \in \mathcal{B}_q^{\mathcal{E}}(T)$.

It remains to be proved that $\sigma_S = \sigma_T$ for any $S, T \in \mathcal{A}$. As $\mathcal{A}$ is a patch, it suffices to show it for $S \cap T \neq \emptyset$. Then we have $S \in \mathcal{B}_1^{\mathcal{E}}(T)$, and therefore $\mathcal{B}_{q-1}^{\mathcal{E}}(S) \subset \mathcal{B}_q^{\mathcal{E}}(T)$. Consequently, for each $U \in \mathcal{B}_{q-1}^{\mathcal{E}}(S)$, we have $U\sigma_S = Uf$ since $U \in \mathcal{B}_q^{\mathcal{E}}(S)$, and $U\sigma_T = Uf$ since $U \in \mathcal{B}_q^{\mathcal{E}}(T)$. It follows $U\sigma_S = U\sigma_T$ for each $U \in \mathcal{B}_{q-1}^{\mathcal{E}}(S)$, and therefore $\sigma_S = \sigma_T$. ■

Corollary 1.4. For each integer k, there are finitely many isomorphism classes of Δ-patches consisting of k tiles.

Proof. This result follows from Theorem 1.3 since there are finitely many isomorphism classes of $\mathcal{L}_\Delta$-structures consisting of k elements. ■

Corollary 1.5. For each finite Δ-patch $\mathcal{E}$ and each $k \in \mathbb{N}^*$, finitely many distinct Δ-patches can be obtained from $\mathcal{E}$ by adding k new tiles.

Proof. Denote by Ω the set of all Δ-patches obtained from $\mathcal{E}$ by adding k new tiles. By corollary 1.4, it suffices to show that, for each $\mathcal{F} \in \Omega$, there

exist finitely many pairs $(\mathcal{G}, \sigma)$ with $\mathcal{G} \in \Omega$ and $\sigma : \mathcal{F} \to \mathcal{G}$ isomorphism. But any such pair is completely determined by the tiles $\sigma^{-1}(T)$ for $T \in \mathcal{E}$, since no $\rho \in G - \{\mathrm{Id}\}$ fixes the tiles of $\mathcal{E}$. ■

Corollary 1.6. For each $k \in \mathbb{N}$, $\{(\mathcal{B}_k^{\mathcal{E}}(T), T) \mid \mathcal{E}$ satisfies Δ and $T \in \mathcal{E}\}$ is a finite union of isomorphism classes.

Proof. For $k \leq q$, this property is true because $\mathcal{B}_k^{\mathcal{E}}(T)$ is contained in $\mathcal{B}_q^{\mathcal{E}}(T)$ for each set $\mathcal{E}$ of tiles which satisfies Δ and each $T \in \mathcal{E}$. For $k \geq q+1$, it follows from Corollary 1.4 since we have a finite bound for the cardinals of the Δ-patches $\mathcal{B}_k^{\mathcal{E}}(T)$ for $\mathcal{E}$ satisfying Δ and $T \in \mathcal{E}$. ■

Corollary 1.7. Any set of tiles which satisfies Δ is a tiling.

Proof. Any such set $\mathcal{E}$ covers E by the first part of Proposition 1.2. Moreover, for each $k \in \mathbb{N}$, $\{(\mathcal{B}_k^{\mathcal{E}}(T), T) \mid T \in \mathcal{E}\}$ is a finite union of isomorphism classes by Corollary 1.6. ■

Remark. The following variant of Example 1 above shows that, in Theorem 1.3 and its corollaries, it is not enough to suppose that each $\mathcal{D}_i$ is fixed by no $\sigma \in G - \{\mathrm{Id}\}$. Consider the coverings $\mathcal{E}$ of $\mathbb{R}^2$ obtained from the following prototiles:
$T_0 = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } x^2 + y^2 \leq 1\}$,
$T_k = \{(x, y) \in \mathbb{R}^2 \mid k^2 \leq x^2 + y^2 \leq (k+1)^2\}$ for $1 \leq k \leq 2q$,
$T_{2q+1} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq (2q+1)^2 \text{ and } \sup(|x|, |y|) \leq 2q+2\}$,
with some bumps on the diameter of T_0 and on the four sides of T_{2q+1} so that T_0 and T_{2q+1} are fixed by no $\sigma \in G - \{\mathrm{Id}\}$. Then, in each such $\mathcal{E}$, each $\mathcal{B}_q^{\mathcal{E}}(T)$ is fixed by no $\sigma \in G - \{\mathrm{Id}\}$, but the pairs $(\mathcal{B}_{q+1}^{\mathcal{E}}(T), T)$ for $T \in \mathcal{E}$ do not generally fall in finitely many isomorphism classes.

We say that a group H of isometries of E is *discrete* if, for each $x \in E$, there exists $\eta \in \mathbb{R}_{>0}$ such that $\beta(x, \eta) \cap \{x\sigma \mid \sigma \in H\} = \{x\}$. This property is true if and only if $\beta(y, \eta) \cap \{x\sigma \mid \sigma \in H\}$ is finite for each $\eta \in \mathbb{R}_{>0}$ and any $x, y \in E$.

Proposition 1.8. For each tiling $\mathcal{T}$ which satisfies Δ, the subgroup $H = \{\sigma \in G \mid \mathcal{T}\sigma = \mathcal{T}\}$ is discrete.

Proof. We show that $\beta(y, \eta) \cap \{x\sigma \mid \sigma \in H\}$ is finite for each $\eta \in \mathbb{R}_{>0}$ and any $x, y \in E$. We consider $T \in \mathcal{T}$ such that $x \in T$. Any $\sigma \in H$ such that $x\sigma \in \beta(y, \eta)$ sends T to a tile U such that $U \cap \beta(y, \eta) \neq \varnothing$ and induces a bijection from $\mathcal{B}_q^{\mathcal{T}}(T)$ to $\mathcal{B}_q^{\mathcal{T}}(U)$ which completely determines σ. By the first part of Proposition 1.2, $\mathcal{T}$ contains finitely many tiles U such that $U \cap \beta(y, \eta) \neq \varnothing$. Moreover, for each such U, there exist finitely many

bijections from $\mathcal{B}_q^{\mathcal{T}}(T)$ to $\mathcal{B}_q^{\mathcal{T}}(U)$. Consequently, $\{\sigma \in H \mid x\sigma \in \beta(y,\eta)\}$ is finite. ■

Now we fix $\lambda \in \mathbb{R}_{>0}$ and we state some supplementary conditions on (E,δ) and G which imply that each tiling with tiles of radius $< \lambda$ satisfies a local rule with the properties stated above. We suppose that (E,δ) is geodesic and we consider the following properties:
(CVXλ) For each $x \in E$ and each $\eta \in]0,\lambda[$, each geodesic which joins two points of $\beta(x,\eta)$ is contained in $\cup_{\zeta<\eta}\beta(x,\zeta)$;
(FIXλ) There exists $\mu \in \mathbb{R}_{>0}$ such that:
(FIX$\lambda\mu$) for each $x \in E$, no $\sigma \in G - \{\mathrm{Id}\}$ fixes the points of a set $A \subset E$ with $\beta(x,\mu) \subset \cup_{y\in A}\beta(y,\lambda)$.

In euclidean or hyperbolic spaces of finite dimension, (CVXν) and (FIXν) are true for each $\nu \in \mathbb{R}_{>0}$. On the other hand, for the surfaces of a sphere or a cylinder of infinite length, they are only true for ν small enough. We do not want to suppose from the beginning that they are true for each $\nu \in \mathbb{R}_{>0}$ since, for instance, any tiling of $\mathbb{R}^2$ which is invariant through a nontrivial translation induces a tiling of the surface of a cylinder of infinite length.

Lemma 1.9. Suppose that (E,δ) is geodesic and satisfies (CVXλ). Then, for each nonempty $S \subset E$ such that $\mathrm{Rad}(S) < \lambda$, there exists a unique $x \in E$ such that $\delta(x,y) \leq \mathrm{Rad}(S)$ for each $y \in S$, and we have $x\sigma = x$ for each isometry σ such that $S\sigma = S$.

Proof. We only prove the first statement, since the second one is an immediate consequence. For each $w \in E$, we have $\sup_{y\in S}\delta(w,y) = \sup_{y\in T}\delta(w,y)$ where T is the closure of S in E. Consequently, we can suppose S closed, and therefore S compact. The subset $A = \{w \in E \mid \sup_{y\in S}\delta(w,y) \leq \lambda\}$ is closed, and therefore compact since it is contained in $\beta(y,\lambda)$ for each $y \in S$. Consequently, there exists $x \in A$ such that $\sup_{y\in S}\delta(x,y) = \inf_{w\in A}(\sup_{y\in S}\delta(w,y)) = \mathrm{Rad}(S)$.

Now suppose that there exists $x' \neq x$ in E such that $\sup_{y\in S}\delta(x',y) = \mathrm{Rad}(S)$, and consider $x'' \in E$ such that $\delta(x,x'') = \delta(x'',x') = \delta(x,x')/2$. By (CVX$\lambda$), we have $\delta(x'',y) < \sup(\delta(x,y),\delta(x',y)) \leq \mathrm{Rad}(S)$ for each $y \in S$. It follows $\sup_{y\in S}\delta(x'',y) \leq \mathrm{Rad}(S)$, and therefore $\sup_{y\in S}\delta(x'',y) = \mathrm{Rad}(S)$. As S is compact, there exists $y \in S$ such that $\delta(x'',y) = \mathrm{Rad}(S)$, which contradicts (CVX$\lambda$) since $\delta(x,y) \leq \mathrm{Rad}(S)$ and $\delta(x',y) \leq \mathrm{Rad}(S)$. ■

Proposition 1.10. Suppose that (E,δ) is geodesic and satisfies (CVXλ). Suppose that (FIX$\lambda\mu$) is true for some $\mu \in \mathbb{R}_{>0}$. Let $\Gamma = \{(\mathcal{C}_1,C_1),...,(\mathcal{C}_m,C_m)\}$ be a 1-local rule defined with tiles of radius $< \lambda$. Consider $\xi \in \mathbb{R}_{>0}$ such that, for each $i \in \{1,...,m\}$, the union of the tiles of $\mathcal{C}_i$ contains $\cup_{x\in C_i}\beta(x,\xi)$, and $p \in \mathbb{N}^*$ such that $p\xi \geq \mu$. Then no $\sigma \in G - \{\mathrm{Id}\}$ stabilizes the tiles of a

p-configuration $(\mathcal{D}, D)$ which is compatible with Γ. In particular, Theorem 1.3 and its corollaries are true for each $(p+1)$-local rule Δ which is compatible with Γ.

Proof. Suppose that some $\sigma \in G - \{\mathrm{Id}\}$ stabilizes the tiles of a p-configuration $(\mathcal{D}, D)$ which is compatible with Γ. As (E, δ) is geodesic and as each $(\mathcal{B}_1^{\mathcal{D}}(T), T)$ with $T \in \mathcal{B}_{p-1}^{\mathcal{D}}(D)$ is isomorphic to some $(\mathcal{C}_i, C_i)$, we see by induction on $0 \leq k \leq p$ that $\beta(x, k\xi)$ is contained in the union of the tiles of $\mathcal{B}_k^{\mathcal{D}}(D)$. Consequently, $\beta(x, \mu) \subset \beta(x, p\xi)$ is contained in the union of the tiles of $\mathcal{D}$. For each $y \in \beta(x, \mu)$ and each $T \in \mathcal{D}$ such that $y \in T$, we have $\delta(y, x_T) \leq \mathrm{Rad}(T) < \lambda$, where x_T is the unique point of E such that $\delta(z, x_T) \leq \mathrm{Rad}(T)$ for each $z \in T$. This contradicts (FIX$\lambda\mu$) since $x_T\sigma = x_T$ for each $T \in \mathcal{D}$ by Lemma 1.9. ■

2. Local isomorphism and representation of relational structures by tilings.

First we define local isomorphism and the extraction preorder $\Subset$ for tilings and relational structures. The definitions for tilings are classical. We note that, by the first part of Proposition 1.2, each subpatch of a tiling is finite if and only if it is bounded.

We say that a tiling $\mathcal{T}$ satisfies the *local isomorphism property* if, for each finite subpatch $\mathcal{E}$ of $\mathcal{T}$, there exists $k \in \mathbb{N}^*$ such that each $\mathcal{B}_k^{\mathcal{T}}(T)$ contains a copy of $\mathcal{E}$. Then, for each finite subpatch $\mathcal{E}$ of $\mathcal{T}$, there exists $\rho \in \mathbb{R}_{>0}$ such that each $\beta(x, \rho) \subset E$ contains a subpatch of $\mathcal{T}$ which is isomorphic to $\mathcal{E}$. The second part of Proposition 1.2 implies that the converse is true if E is weakly homogeneous.

For any tilings $\mathcal{S}, \mathcal{T}$, we write $\mathcal{S} \Subset \mathcal{T}$ if each finite subpatch of $\mathcal{S}$ is isomorphic to a subpatch of $\mathcal{T}$. We say that $\mathcal{S}$ and $\mathcal{T}$ are *locally isomorphic* if $\mathcal{S} \Subset \mathcal{T}$ and $\mathcal{T} \Subset \mathcal{S}$.

Now we give the definitions and the notations for relational structures, which are similar to those in [7, pp. 107, 112, 113]. We consider a finite relational language $\mathcal{L}$.

For each $\mathcal{L}$-structure M and each $u \in M$, we define inductively the subsets $B_M(u, h)$ with $B_M(u, 0) = \{u\}$ and, for $h \in \mathbb{N}$,
$B_M(u, h+1) = B_M(u, h) \cup \{v \in M \mid$ there exist $R(x_1, ..., x_k) \in \mathcal{L}$, $u_1, ..., u_k \in M$ and $1 \leq i, j \leq k$ such that $R(u_1, ..., u_k)$, $u_i \in B_M(u, h)$ and $u_j = v\}$.
We say that M is *connected* if there exists $u \in M$ such that $M = \cup_{h \in \mathbb{N}} B_M(u, h)$.

We say that M is *locally finite* if $B_M(u, 1)$ is finite for each $u \in M$. Then $B_M(u, h)$ is finite for any $u \in M$ and $h \in \mathbb{N}$. We say that M is *uniformly locally finite* if there exists $r \in \mathbb{N}^*$ such that $|B_M(u, 1)| \leq r$ for each $u \in M$. Then, for each $h \in \mathbb{N}$, there exists $s \in \mathbb{N}^*$ such that $|B_M(u, h)| \leq s$ for each $u \in M$. We say that M satisfies the *local isomorphism property* if, for any

$u \in M$ and $h \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that each $B_M(v,k)$ contains some w with $(B_M(w,h),w) \cong (B_M(u,h),u)$.

For any $\mathcal{L}$-structures M, N, we write $M \Subset N$ if, for any $u \in M$ and $h \in \mathbb{N}$, $|\{v \in M \mid (B_M(v,h),v) \cong (B_M(u,h),u)\}| \leq$
$|\{w \in N \mid (B_N(w,h),w) \cong (B_M(u,h),u)\}|$
or both sets are infinite. For M, N connected, we have $M \Subset N$ if and only if, for any $u \in M$ and $h \in \mathbb{N}$, there exists $v \in N$ such that $(B_M(u,h),u) \cong (B_N(v,h),v)$.

We say that M and N are *locally isomorphic* if $M \Subset N$ and $N \Subset M$. By [7, Theorem 2.3], two locally finite $\mathcal{L}$-structures M, N are elementarily equivalent if and only if they are locally isomorphic.

Now we consider again the metric space with bounded closed subsets (E,δ), the group G of bijective isometries of E, the integer q, the q-local rule $\Delta = \{(\mathcal{D}_1, D_1), ..., (\mathcal{D}_n, D_n)\}$ such that each $\mathcal{B}^{\mathcal{D}_i}_{q-1}(D_i)$ is fixed by no $\sigma \in G - \{\mathrm{Id}\}$, and the language $\mathcal{L}_\Delta$, which were introduced for Theorem 1.3. We call a Δ-*tiling* any tiling which satisfies Δ.

We define a *representation* of a $\mathcal{L}_\Delta$-structure M as a pair $(\mathcal{E}, f)$, where $\mathcal{E}$ is a Δ-patch and $f : M \to \mathcal{E}$ is an isomorphism of $\mathcal{L}_\Delta$-structures. By Theorem 1.3, the representation is unique up to isomorphism if it exists.

For each $\mathcal{L}_\Delta$-structure M and each $u \in M$, we define inductively the subsets $B^M_h(u)$ with $B^M_0(u) = \{u\}$ and, for each $h \in \mathbb{N}$,
$B^M_{h+1}(u) = B^M_h(u) \cup \{v \in M \mid$ there exist $1 \leq i \leq n$, $v_1, ..., v_{p(i)} \in M$ and $1 \leq j,k \leq p(i)$ such that $R_i(v_1, ..., v_{p(i)})$, $v_j \in B^M_h(u)$, $v_k = v$ and $D_{i,j} \cap D_{i,k} \neq \emptyset\}$
where the sets $D_{i,j}$ are the tiles used for the definition of $\mathcal{L}_\Delta$.

For each set $\mathcal{E}$ of tiles, each $T \in \mathcal{E}$ and each $h \in \mathbb{N}$, the $\mathcal{L}_\Delta$-structure M defined on $\mathcal{E}$ satisfies $B^M_h(T) \subset \mathcal{B}^{\mathcal{E}}_h(T)$, and $B^M_h(T) = \mathcal{B}^{\mathcal{E}}_h(T)$ if $\mathcal{E}$ is a Δ-tiling. The notation $B^M_h(u)$ should not be confused with $B_M(u,h)$. For each $\mathcal{L}_\Delta$-structure M, each $u \in M$ and each $h \in \mathbb{N}$, we have $B^M_h(u) \subset B_M(u,h) \subset B^M_{2qh}(u)$, and $B_M(u,h) = B^M_{2qh}(u)$ if M can be represented by a Δ-tiling.

Now we introduce something analogous to the notion of local rule considered for tilings. A *local rule* for a finite relational language $\mathcal{L}$ is defined by specifying an integer $r \in \mathbb{N}^*$ and a finite sequence of pairs $(M_1, x_1), ..., (M_k, x_k)$ with $M_1, ..., M_k$ finite $\mathcal{L}$-structures and $M_i = B_{M_i}(x_i, r)$ for $1 \leq i \leq k$. We say that a $\mathcal{L}$-structure M *satisfies* that rule if each $(B_M(x,r),x)$ is isomorphic to one of the pairs (M_i, x_i).

Any such rule can be expressed by a first-order sentence. For each $s \in \mathbb{N}^*$, the pairs $(B_M(x,s),x)$ for M satisfying that rule and $x \in M$ are finite and fall in finitely many isomorphism classes. In particular, any $\mathcal{L}$-structure which satisfies a local rule is uniformly locally finite.

Theorem 2.1 below implies that there exists a generally infinite set of local rules which characterizes, among the connected $\mathcal{L}_\Delta$-structures, those

which can be represented by Δ-tilings. The problem of the characterization of representable $\mathcal{L}_\Delta$-structures by a finite set of local rules will be considered with Theorem 2.7 and the examples in Section 4.

The following property of a $\mathcal{L}_\Delta$-structure M is true, in particular, if M can be represented by a Δ-tiling:
(P) For each $u \in M$, there exist $1 \leq i \leq n$ and $u_2, ..., u_{p(i)} \in M$ such that $R_i(u, u_2, ..., u_{p(i)})$.

Theorem 2.1. For each connected $\mathcal{L}_\Delta$-structure M, the following properties are equivalent:
1) M can be represented by a Δ-tiling;
2) For each $u \in M$ and each $r \in \mathbb{N}^*$, we have $(B_M(u,r), u) \cong (B_\mathcal{T}(T,r), T)$ for a tile T of a Δ-tiling $\mathcal{T}$;
3) M satisfies (P) and, for each $u \in M$ and each integer $h \geq q$, there exists a representation of $B_h^M(u)$ by a Δ-patch.

Proof. The property 1) clearly implies 2).

Suppose that 2) is true. Consider any element $u \in M$ and any integer $h \geq q$. Then, for each Δ-tiling $\mathcal{T}$ and each $T \in \mathcal{T}$, each isomorphism from $(B_M(u,h), u)$ to $(B_\mathcal{T}(T,h), T)$ induces an isomorphism from $(B_h^M(u), u) \subset (B_M(u,h), u)$ to $(B_h^\mathcal{T}(T), T)$. Such an isomorphism gives a representation of $B_h^M(u)$ since $B_h^\mathcal{T}(T) = \mathcal{B}_h^\mathcal{T}(T)$ is a Δ-patch. It also induces an isomorphism from $(B_q^M(u), u)$ to $(B_q^\mathcal{T}(T), T)$, which implies that there exist $1 \leq i \leq n$ and $u_2, ..., u_{p(i)} \in M$ such that $R_i(u, u_2, ..., u_{p(i)})$.

Now suppose that 3) is true and fix $u \in M$. For each integer $h \geq q$, consider a representation $(\mathcal{E}_h, f_h)$ of $B_h^M(u)$ by a Δ-patch. For $k \geq h \geq q$, as $f_h^{-1} f_k$ is a homomorphism of $\mathcal{L}_\Delta$-structures, Theorem 1.3 implies that there exists a unique $\sigma_{h,k} \in G$ such that $S f_h^{-1} f_k = S\sigma_{h,k}$ for each $S \in \mathcal{E}_h$. So we can suppose that, for $k \geq h \geq q$, we have $\mathcal{E}_h \subset \mathcal{E}_k$ and f_h is the restriction of f_k to $B_h^M(u)$. Then $f = \cup_{h \geq q} f_h$ is a $\mathcal{L}_\Delta$-isomorphism from M to $\mathcal{E} = \cup_{h \geq q} \mathcal{E}_h$. As M satisfies (P), it follows that $\mathcal{E}$ satisfies Δ, and $\mathcal{E}$ is a tiling by Corollary 1.7. ■

Corollary 2.2. For any connected $\mathcal{L}_\Delta$-structures $M \Subset N$, if N can be represented by a Δ-tiling, then M can also be represented by a Δ-tiling.

Similarly to [7], we have:

Proposition 2.3. 1) Any Δ-tiling satisfies the local isomorphism property as a tiling if and only if it satisfies the local isomorphism property as a $\mathcal{L}_\Delta$-structure.
2) Two Δ-tilings $\mathcal{S}, \mathcal{T}$ satisfy $\mathcal{S} \Subset \mathcal{T}$ (resp. are locally isomorphic) as tilings if and only if they satisfy $\mathcal{S} \Subset \mathcal{T}$ (resp. are locally isomorphic) as $\mathcal{L}_\Delta$-structures.

Proof. The following facts will be used in the proofs of 1) and 2):
a) We have $B_{\mathcal{T}}(T,h) = B^{\mathcal{T}}_{2qh}(T) = \mathcal{B}^{\mathcal{T}}_{2qh}(T)$ for each Δ-tiling $\mathcal{T}$, each $T \in \mathcal{T}$ and each $h \in \mathbb{N}$.
b) Theorem 1.3 implies that, for any Δ-tilings $\mathcal{S}, \mathcal{T}$, each $S \in \mathcal{S}$, each $T \in \mathcal{T}$ and each integer $k \geq q$, the Δ-patches $\mathcal{B}^{\mathcal{S}}_k(S)$ and $\mathcal{B}^{\mathcal{T}}_k(T)$ are isomorphic as sets of tiles if and only if they are isomorphic as $\mathcal{L}_\Delta$-structures.
c) For any Δ-tilings $\mathcal{S}, \mathcal{T}$, each $S \in \mathcal{S}$, each $k \in \mathbb{N}^*$ and each $\sigma \in G$ such that $\mathcal{B}^{\mathcal{S}}_k(S)\sigma \subset \mathcal{T}$, we have $\mathcal{B}^{\mathcal{S}}_k(S)\sigma = \mathcal{B}^{\mathcal{T}}_k(S\sigma)$.
d) Each finite subpatch of a Δ-tiling $\mathcal{T}$ is contained in some $\mathcal{B}^{\mathcal{T}}_h(T)$.

First we prove 1). The facts a), b), c) above imply that $\mathcal{T}$ satisfies the local isomorphism property as a $\mathcal{L}_\Delta$-structure if and only if, for each $h \in \mathbb{N}^*$ and each $S \in \mathcal{T}$, there exists $k \in \mathbb{N}^*$ such that each $\mathcal{B}^{\mathcal{T}}_k(T)$ contains a copy of $\mathcal{B}^{\mathcal{T}}_h(S)$. By d), the last property is true if and only if $\mathcal{T}$ satisfies the local isomorphism property as a tiling.

Now we prove 2). We only show the first statement, since the second one is an immediate consequence. The facts a), b), c) above imply that $\mathcal{S}$ and $\mathcal{T}$ satisfy $\mathcal{S} \Subset \mathcal{T}$ as $\mathcal{L}_\Delta$-structures if and only if $\mathcal{T}$ contains a copy of $\mathcal{B}^{\mathcal{S}}_h(S)$ for each $h \in \mathbb{N}^*$ and each $S \in \mathcal{S}$. By d), the last property is true if and only if $\mathcal{S}$ and $\mathcal{T}$ satisfy $\mathcal{S} \Subset \mathcal{T}$ as tilings. ■

By Theorem 2.1, for each sequence $(B_{M_i}(u_i,r_i),u_i)_{i\in\mathbb{N}} = (B^{M_i}_{2qr_i}(u_i),u_i)_{i\in\mathbb{N}}$ of pairs taken in $\mathcal{L}_\Delta$-structures associated to Δ-tilings, with $r_i < r_j$ for $i < j$, the inductive limit relative to any sequence of isomorphisms
$f_i : (B_{M_i}(u_i,r_i),u_i) \to (B_{M_{i+1}}(u_{i+1},r_i),u_{i+1}) \subset (B_{M_{i+1}}(u_{i+1},r_{i+1}),u_{i+1})$
is a pair (M,x) with M a $\mathcal{L}_\Delta$-structure associated to a Δ-tiling. Using this fact, together with Corollary 2.2 and Proposition 2.3, we see that Corollary 2.4 (resp. 2.5, resp. 2.6) below, in the same way as [7, Corollary 3.5] (resp. [7, Corollary 3.6], resp. [7, Corollary 3.7]) is a consequence of [7, Corollary 3.2] (resp. [7, Proposition 3.3], resp. [7, Proposition 3.4]) and its proof.

Corollary 2.4. Any Δ-tiling is minimal for $\Subset$ if and only if it satisfies the local isomorphism property.

Corollary 2.5. For each Δ-tiling $\mathcal{S}$, there exists a Δ-tiling $\mathcal{T} \Subset \mathcal{S}$ which is minimal for $\Subset$.

Corollary 2.6. For each Δ-tiling $\mathcal{S}$, we have:
1) If there are finitely many equivalence classes of elements of $\mathcal{S}$ modulo the isometries $\sigma \in G$ such that $\mathcal{S}\sigma = \mathcal{S}$, then any Δ-tiling $\mathcal{T} \Subset \mathcal{S}$ is isomorphic to $\mathcal{S}$.
2) If $\mathcal{S}$ is minimal for $\Subset$, and if there are infinitely many equivalence classes of elements of $\mathcal{S}$ modulo the isometries $\sigma \in G$ such that $\mathcal{S}\sigma = \mathcal{S}$, then there exist 2^ω pairwise nonisomorphic Δ-tilings which are locally isomorphic to $\mathcal{S}$.

Remark. Corollary 2.6 implies [11, Theorem, p. 356]. In [11], Radin and Wolff considered tilings of the euclidean spaces $\mathbb{R}^n$, and isomorphism was defined modulo an arbitrary group of isometries.

Theorem 2.7 below is analogous to [7, Theorem 5.2]. Here, we suppose that (E, δ) is weakly homogeneous and that, for any $x, y \in E$ and each $\eta \in \mathbb{R}_{>0}$, $\beta(x, \delta(x, y)) \cap \beta(y, \eta)$ is connected and contains a point z with $\delta(x, z) < \delta(x, y)$.

These conditions are true for each geodesic space which satisfies the properties (CVXν) considered at the end of Section 1. In each $\mathbb{R}^k$, they are also true for the distances defined with
$\delta((x_1, ..., x_k), (y_1, ..., y_k)) = \sup(|y_1 - x_1|, ..., |y_k - x_k|)$ or
$\delta((x_1, ..., x_k), (y_1, ..., y_k)) = |y_1 - x_1| + ... + |y_k - x_k|)$.

On the other hand, the connectedness condition is not true if E is the surface of a cylinder of infinite length and if δ is defined by considering E as a quotient of the euclidean space $\mathbb{R}^2$. Actually, Theorem 2.7 is not true in that case since, for some local rules Δ, it is possible to find a $\mathcal{L}_\Delta$-structure M which can only be represented by a tiling of $\mathbb{R}^2$, while the substructures $B_r^M(u)$ considered in Theorem 2.7 are small enough to be represented in (E, δ).

Again, we consider $\mathcal{L}_\Delta$-structures which satisfy the property (P) introduced for Theorem 2.1. We denote by λ the maximum of the diameters of the tiles in Δ, and ξ the largest real number such that, for each $i \in \{1, ..., n\}$, the union of the tiles of $\mathcal{B}_1^{\mathcal{D}_i}(D_i)$ contains $\cup_{x \in D_i} \beta(x, \xi)$. For each $r \in \mathbb{N}$, we write $\omega(\xi, r) = \inf_{x \in E} \omega(x, \xi, r)$, as in the definition of weakly homogeneous spaces.

Theorem 2.7. Suppose that (E, δ) is weakly homogeneous and that, for any $x, y \in E$ and each $\eta \in \mathbb{R}_{>0}$, $\beta(x, \delta(x, y)) \cap \beta(y, \eta)$ is connected and contains a point z with $\delta(x, z) < \delta(x, y)$. Define (P), λ and ξ as above. Consider an integer $r \geq q + 1$ such that $\omega(\xi, r - q - 1) \geq (4q + 1)\lambda$. Let M be a connected $\mathcal{L}_\Delta$-structure which satisfies (P) and such that each $B_r^M(u)$ can be represented by a Δ-patch. Then there exist a Δ-tiling $\mathcal{T}$ and a surjective $\mathcal{L}_\Delta$-homomorphism $\varphi : \mathcal{T} \to M$ which induces a $\mathcal{L}_\Delta$-isomorphism from $\mathcal{B}_r^{\mathcal{T}}(T)$ to $B_r^M(T\varphi)$ for each $T \in \mathcal{T}$.

Remark. The pair $(\mathcal{T}, \varphi)$ satisfies $S\varphi \neq T\varphi$ for each $S \in \mathcal{T}$ and each $T \in \mathcal{B}_{2r}^{\mathcal{T}}(S) - \{S\}$, since there exists $U \in \mathcal{T}$ such that $S, T \in \mathcal{B}_r^{\mathcal{T}}(U)$.

Remark. If (E, δ) is geodesic, we can take any integer $r \geq q+1+(4q+1)\lambda/\xi$.

Remark. It follows that there exists a Δ-tiling if there exists a $\mathcal{L}_\Delta$-structure M which satisfies (P) and such that each $B_r^M(u)$ can be represented by a Δ-patch.

Lemma 2.7.1. For each $u \in M$, each representation $(\mathcal{E}, f)$ of $B_r^M(u)$ and each $x \in uf$, any $T \in \mathcal{E}$ such that $T \cap \beta(x,(4q+1)\lambda) \neq \emptyset$ belongs to $\mathrm{I}_\Delta(\mathcal{E})$.

Proof of Lemma 2.7.1. As M satisfies (P), for each $v \in B_{r-q}^M(u)$, we have $vf \in \mathrm{I}_\Delta(\mathcal{E})$ and $\cup_{y \in vf} \beta(y, \xi)$ is contained in the union of the tiles of $B_1^M(v)f$. An induction on k shows that $\beta(x, \omega(\xi, k))$ is contained in the union of the tiles of $B_k^M(u)f$ for $0 \leq k \leq r-q$. In particular, $\beta(x,(4q+1)\lambda) \subset \beta(x, \omega(\xi, r-q-1))$ is contained in the union of the tiles of $B_{r-q-1}^M(u)f$. Consequently, any $T \in \mathcal{E}$ such that $T \cap \beta(x,(4q+1)\lambda) \neq \emptyset$ belongs to $B_{r-q}^M(u)f$, and therefore belongs to $\mathrm{I}_\Delta(\mathcal{E})$. ■

Proof of Theorem 2.7. We consider an element $u \in M$, a representation $(\mathcal{G}, g)$ of $B_r^M(u)$ and a point $x \in ug$. We denote by Ω the set of all pairs $(\mathcal{E}, \varphi)$ such that:

a) $\mathcal{E}$ is a finite Δ-patch, x belongs to a tile of $\mathcal{E}$, any $T \in \mathcal{E}$ such that $x \in T$ belongs to $\mathrm{I}_\Delta(\mathcal{E})$, and φ is a $\mathcal{L}_\Delta$-homomorphism from $\mathcal{E}$ to M;

b) for each $S \in \mathcal{E}$, there exists $T \in \mathrm{I}_\Delta(\mathcal{E})$ such that $S \in \mathcal{B}_q^{\mathcal{E}}(T)$ and $T \cap \beta(x, \rho_{\mathcal{E}}) \neq \emptyset$, where $\rho_{\mathcal{E}} = \sup\{\rho \in \mathbb{R}_{\geq 0} \mid$ any $U \in \mathcal{E}$ such that $U \cap \beta(x, \rho) \neq \emptyset$ belongs to $\mathrm{I}_\Delta(\mathcal{E})\}$.

For each $(\mathcal{E}, \varphi) \in \Omega$, $\beta(x, \rho_{\mathcal{E}})$ is covered by the tiles of $\mathcal{E}$: For each $y \in E$ such that $\delta(x, y) = \rho_{\mathcal{E}}$, the second condition on (E, δ) implies that y is the limit of a sequence $(y_k)_{k \in \mathbb{N}} \subset \cup_{\zeta < \rho_{\mathcal{E}}} \beta(x, \zeta)$. As $\mathrm{I}_\Delta(\mathcal{E})$ is finite, infinitely many y_k belong to the same $T \in \mathrm{I}_\Delta(\mathcal{E})$ and T also contains y.

On the other hand, there exists $y \in E$ with $\delta(x, y) = \rho_{\mathcal{E}}$ which belongs to some $S \in \mathcal{E} - \mathrm{I}_\Delta(\mathcal{E})$: For any sequences $(y_k)_{k \in \mathbb{N}} \subset E$ and $(S_k)_{k \in \mathbb{N}} \subset \mathcal{E} - \mathrm{I}_\Delta(\mathcal{E})$ such that $y_k \in S_k$ for each $k \in \mathbb{N}$ and $\lim_{k \to +\infty} \delta(y_k, \beta(x, \rho_{\mathcal{E}})) = 0$, there exists a subsequence of $(y_k)_{k \in \mathbb{N}}$ which converges to a point $y \in \beta(x, \rho_{\mathcal{E}})$. As $\mathcal{E} - \mathrm{I}_\Delta(\mathcal{E})$ is finite, infinitely many y_k in this subsequence belong to the same S_h and S_h also contains y.

By Lemma 2.7.1, Ω contains $(\mathcal{E}_0, \varphi_0)$, where $\mathcal{E}_0$ is the union of the subsets $\mathcal{B}_q^{\mathcal{G}}(T)$ for $T \in \mathcal{G}$ such that $T \cap \beta(x,(4q+1)\lambda) \neq \emptyset$, and φ_0 is the restriction of g^{-1} to $\mathcal{E}_0$. We have $\rho_{\mathcal{E}_0} > (4q+1)\lambda$.

For any $(\mathcal{E}, \varphi), (\mathcal{F}, \psi) \in \Omega$, we write $(\mathcal{E}, \varphi) \leq (\mathcal{F}, \psi)$ if $\mathcal{E} \subset \mathcal{F}$ and if φ is the restriction of ψ to $\mathcal{E}$. It suffices to prove the two following claims:

1. *The conclusion of Theorem 2.7 is true if Ω contains some strictly increasing $(\mathcal{E}_i, \varphi_i)_{i \in \mathbb{N}}$.*

First we observe that, for each $i \in \mathbb{N}$, there exists $T \in \mathrm{I}_\Delta(\mathcal{E}_{i+1}) - \mathrm{I}_\Delta(\mathcal{E}_i)$ such that $T \cap \beta(x, \rho_{\mathcal{E}_i}) \neq \emptyset$: If $\rho_{\mathcal{E}_{i+1}} > \rho_{\mathcal{E}_i}$, then any $T \in \mathcal{E}_i - \mathrm{I}_\Delta(\mathcal{E}_i)$ such that $T \cap \beta(x, \rho_{\mathcal{E}_i}) \neq \emptyset$ belongs to $\mathrm{I}_\Delta(\mathcal{E}_{i+1})$. If $\rho_{\mathcal{E}_{i+1}} = \rho_{\mathcal{E}_i}$, then, for each $S \in \mathcal{E}_{i+1} - \mathcal{E}_i$, there exists $T \in \mathrm{I}_\Delta(\mathcal{E}_{i+1})$ such that $S \in \mathcal{B}_q^{\mathcal{E}_{i+1}}(T)$ and $T \cap \beta(x, \rho_{\mathcal{E}_i}) = \emptyset$; any such T cannot belong to $\mathrm{I}_\Delta(\mathcal{E}_i)$ since $\mathcal{B}_q^{\mathcal{E}_{i+1}}(T) - \mathcal{B}_q^{\mathcal{E}_i}(T) \neq \emptyset$.

Then we use the function $\omega(\xi,n)$ in order to prove that $\rho_{\mathcal{E}_i}$ tends to infinity. As (E,δ) is weakly homogeneous, $\omega(\xi,n)$ tends to infinity with n. Consequently, it suffices to show that, for any $i,n\in\mathbb{N}$ such that $\rho_{\mathcal{E}_i}>\omega(\xi,n)$, there exists $j>i$ such that $\rho_{\mathcal{E}_j}>\omega(\xi,n+1)$.

For each $j\geq i$ such that $\rho_{\mathcal{E}_j}\leq\omega(\xi,n+1)$, we consider $T_j\in \mathrm{I}_\Delta(\mathcal{E}_{j+1})-\mathrm{I}_\Delta(\mathcal{E}_j)$ such that $T_j\cap\beta(x,\rho_{\mathcal{E}_j})\neq\emptyset$, and $y_j\in T_j\cap\beta(x,\rho_{\mathcal{E}_j})$. There exist $z_j\in\beta(x,\omega(\xi,n))$ such that $\delta(y_j,z_j)\leq\xi$, and $U_j\in\mathcal{B}_1^{\mathcal{E}j}(T_j)$ such that $z_j\in U_j$. We have $U_j\in\mathrm{I}_\Delta(\mathcal{E}_i)$ since $z_j\in\beta(x,\omega(\xi,n))$ and $\omega(\xi,n)<\rho_{\mathcal{E}_i}$. It follows $T_j\in\mathcal{E}_i$. As $\mathcal{E}_i$ is finite, there exist at most finitely many such T_j, and therefore finitely many integers $j>i$ such that $\rho_{\mathcal{E}_j}\leq\omega(\xi,n+1)$.

As $\rho_{\mathcal{E}_i}$ tends to infinity, $\mathcal{T}=\cup_{i\in\mathbb{N}}\mathcal{E}_i$ satisfies Δ, and $\mathcal{T}$ is a Δ-tiling by Corollary 1.7. The inductive limit φ of the maps φ_i is a $\mathcal{L}_\Delta$-homomorphism from $\mathcal{T}$ to M. In particular, we have $\mathcal{B}_k^{\mathcal{T}}(T)\varphi\subset B_k^M(T\varphi)$ for each $T\in\mathcal{T}$ and each $k\in\mathbb{N}$.

Now we show that, for each $T\in\mathcal{T}$, φ induces a $\mathcal{L}_\Delta$-isomorphism from $\mathcal{B}_r^{\mathcal{T}}(T)$ to $B_r^M(T\varphi)$; it follows that φ is surjective since M is connected.

We consider a representation $(\mathcal{H},h)$ of $B_r^M(T\varphi)$. As $\mathcal{B}_r^{\mathcal{T}}(T)$ and $\mathcal{H}$ are Δ-patches, Theorem 1.3 implies that there exists $\sigma\in G$ such that $S\sigma=S\varphi h$ for each $S\in\mathcal{B}_r^{\mathcal{T}}(T)$. As $\mathcal{T}$ is a Δ-tiling and $\mathcal{H}$ is a Δ-patch, σ induces an isomorphism from $\mathcal{B}_q^{\mathcal{T}}(U)$ to $\mathcal{B}_q^{\mathcal{H}}(U\sigma)$ for each $U\in\mathcal{B}_{r-q}^{\mathcal{T}}(T)\subset I_\Delta(\mathcal{B}_r^{\mathcal{T}}(T))$. Consequently, we have $\mathcal{B}_r^{\mathcal{T}}(T)\sigma=\mathcal{B}_r^{\mathcal{H}}(T\sigma)=\mathcal{B}_r^{\mathcal{H}}(T\varphi h)=B_r^M(T\varphi)h=\mathcal{H}$. It follows that σ induces an isomorphism from $\mathcal{B}_r^{\mathcal{T}}(T)$ to $\mathcal{H}$ and φ induces a $\mathcal{L}_\Delta$-isomorphism from $\mathcal{B}_r^{\mathcal{T}}(T)$ to $B_r^M(T\varphi)$.

2. *For each $(\mathcal{E},\varphi)\in\Omega$, there exists $(\mathcal{F},\psi)>(\mathcal{E},\varphi)$ in Ω.*

We consider $y\in E$ with $\delta(x,y)=\rho_{\mathcal{E}}$ which belongs to some $S\in\mathcal{E}-\mathrm{I}_\Delta(\mathcal{E})$, and $T\in\mathrm{I}_\Delta(\mathcal{E})$ such that $y\in T$. There exist a representation of $B_r^M(T\varphi)$, and therefore a Δ-patch $\mathcal{E}'$ and a $\mathcal{L}_\Delta$-isomorphism φ' from $\mathcal{E}'$ to $B_r^M(T\varphi)$. We have $\mathcal{B}_q^{\mathcal{E}}(T)\varphi\subset B_q^M(T\varphi)$ since T belongs to $\mathrm{I}_\Delta(\mathcal{E})$, and $B_q^M(T\varphi)\varphi'^{-1}\subset\mathcal{B}_q^{\mathcal{E}'}(T\varphi\varphi'^{-1})$ since φ' is a $\mathcal{L}_\Delta$-isomorphism. The map $\mathcal{B}_q^{\mathcal{E}}(T)\to\mathcal{B}_q^{\mathcal{E}'}(T\varphi\varphi'^{-1}):U\to U\varphi\varphi'^{-1}$ is a $\mathcal{L}_\Delta$-homomorphism. As $\mathcal{B}_q^{\mathcal{E}}(T)$ and $\mathcal{B}_q^{\mathcal{E}'}(T\varphi\varphi'^{-1})$ are Δ-patches, Theorem 1.3 implies that there exists $\sigma\in G$ such that $U\sigma=U\varphi\varphi'^{-1}$ for each $U\in\mathcal{B}_q^{\mathcal{E}}(T)$. So we can suppose for the remainder of the proof that $\mathcal{B}_q^{\mathcal{E}}(T)\subset\mathcal{E}'$ and $U\varphi=U\varphi'$ for each $U\in\mathcal{B}_q^{\mathcal{E}}(T)$.

We denote by $\mathcal{F}$ the union of $\mathcal{E}$ and the subsets $\mathcal{B}_q^{\mathcal{E}'}(U)$ for $U\in\mathcal{E}'$ such that $y\in U$. We consider the map $\psi:\mathcal{F}\to M$ with $U\psi=U\varphi$ for $U\in\mathcal{E}$ and $U\psi=U\varphi'$ for $U\in\mathcal{F}-\mathcal{E}$.

First we prove that, for each $U\in\mathcal{F}$, we have $\mathcal{B}_q^{\mathcal{F}}(U)\subset\mathcal{E}$ and $V\psi=V\varphi$ for each $V\in\mathcal{B}_q^{\mathcal{F}}(U)$, or $\mathcal{B}_q^{\mathcal{F}}(U)\subset\mathcal{E}'$ and $V\psi=V\varphi'$ for each $V\in\mathcal{B}_q^{\mathcal{F}}(U)$.

If $U\cap\beta(y,2q\lambda)=\emptyset$, then the tiles in $\mathcal{B}_q^{\mathcal{F}}(U)$ cannot belong to $\mathcal{F}-\mathcal{E}$ since they contain no point in $\beta(y,q\lambda)$. Consequently, we have $\mathcal{B}_q^{\mathcal{F}}(U)\subset\mathcal{E}$ and $V\psi=V\varphi$ for each $V\in\mathcal{B}_q^{\mathcal{F}}(U)$.

If $U \cap \beta(y, 2q\lambda) \neq \emptyset$, then we prove that $\mathcal{B}_q^{\mathcal{F}}(U) \subset \mathcal{E}'$ and $V\psi = V\varphi'$ for each $V \in \mathcal{B}_q^{\mathcal{F}}(U)$. For each $V \in \mathcal{B}_q^{\mathcal{F}}(U)$, we have $V \cap \beta(y, 3q\lambda) \neq \emptyset$. So it suffices to show that each $V \in \mathcal{E}$ with $V \cap \beta(y, 3q\lambda) \neq \emptyset$ satisfies $V \in \mathcal{E}'$ and $V\varphi = V\varphi'$. We consider $W \in \mathrm{I}_\Delta(\mathcal{E})$ such that $V \in \mathcal{B}_q^{\mathcal{E}}(W)$ and $W \cap \beta(x, \rho_{\mathcal{E}}) \neq \emptyset$. As $W \cap \beta(y, 4q\lambda) \neq \emptyset$, the properties $V \in \mathcal{E}'$ and $V\varphi = V\varphi'$ follow from Lemma 2.7.2 below.

Consequently, ψ is a $\mathcal{L}_\Delta$-homomorphism and, for each $U \in \mathcal{F}$, there exists $i \in \{1, ..., n\}$ such that $(\mathcal{B}_q^{\mathcal{F}}(U), U) \leq (\mathcal{D}_i, D_i)$. Moreover, the definition of $(\mathcal{F}, \psi)$ implies that, for each $V \in \mathcal{F}$, there exists $U \in \mathrm{I}_\Delta(\mathcal{E})$ such that $V \in \mathcal{B}_q^{\mathcal{E}}(U)$ and $U \cap \beta(x, \rho_{\mathcal{E}}) \neq \emptyset$, or $U \in \mathrm{I}_\Delta(\mathcal{E}')$ such that $V \in \mathcal{B}_q^{\mathcal{E}'}(U)$ and $y \in U$; in both cases, we have $U \in \mathrm{I}_\Delta(\mathcal{F})$, $U \cap \beta(x, \rho_{\mathcal{E}}) \neq \emptyset$ and $V \in \mathcal{B}_q^{\mathcal{F}}(U)$. Consequently, $\mathcal{F}$ is a finite Δ-patch and satisfies the conditions of the definition of Ω. ■

Lemma 2.7.2. For each $W \in \mathrm{I}_\Delta(\mathcal{E})$ such that $W \cap \beta(x, \rho_{\mathcal{E}}) \neq \emptyset$ and $W \cap \beta(y, 4q\lambda) \neq \emptyset$, we have $\mathcal{B}_q^{\mathcal{E}}(W) \subset \mathcal{E}'$ and $X\varphi = X\varphi'$ for each $X \in \mathcal{B}_q^{\mathcal{E}}(W)$.

Proof of Lemma 2.7.2. We fix $z \in W \cap \beta(x, \rho_{\mathcal{E}})$. We have $z \in \beta(y, (4q + 1)\lambda)$. The set $A = \beta(x, \rho_{\mathcal{E}}) \cap \beta(y, (4q + 1)\lambda)$ is connected and contained in the union of the tiles of $\mathrm{I}_\Delta(\mathcal{E})$. Consequently, $\mathcal{A} = \{X \in \mathrm{I}_\Delta(\mathcal{E}) \mid X \cap A \neq \emptyset\}$ is connected. Moreover, T and W belong to $\mathcal{A}$ since y and z belong to A.

We are going to prove that, for each $X \in \mathcal{A}$, we have $\mathcal{B}_q^{\mathcal{E}}(X) \subset \mathcal{E}'$ and $Z\varphi = Z\varphi'$ for each $Z \in \mathcal{B}_q^{\mathcal{E}}(X)$. As this property is true for $X = T$, it suffices to show that, if it is true for some $X \in \mathcal{A}$, then it is true for any $Y \in \mathcal{A}$ such that $X \cap Y \neq \emptyset$.

We have $Y \in \mathcal{B}_1^{\mathcal{E}}(X)$, and therefore $\mathcal{B}_{q-1}^{\mathcal{E}}(Y) \subset \mathcal{E}'$ and $Z\varphi = Z\varphi'$ for each $Z \in \mathcal{B}_{q-1}^{\mathcal{E}}(Y)$. Moreover, by Lemma 2.7.1, we have $Y \in \mathrm{I}_\Delta(\mathcal{E}')$ since $Y \cap \beta(y, (4q+1)\lambda) \neq \emptyset$, and therefore $\mathcal{B}_q^{\mathcal{E}'}(Y)\varphi' = B_q^M(Y\varphi') = B_q^M(Y\varphi)$ since φ' is a $\mathcal{L}_\Delta$-isomorphism from $\mathcal{E}'$ to $B_r^M(T\varphi)$. We also have $\mathcal{B}_q^{\mathcal{E}}(Y)\varphi \subset B_q^M(Y\varphi)$ since $Y \in \mathrm{I}_\Delta(\mathcal{E})$. Consequently, $\varphi\varphi'^{-1}$ is defined on $\mathcal{B}_q^{\mathcal{E}}(Y)$ and stabilizes the elements of $\mathcal{B}_{q-1}^{\mathcal{E}}(Y)$. It follows $\mathcal{B}_q^{\mathcal{E}}(Y) = \mathcal{B}_q^{\mathcal{E}'}(Y)$ and $Z\varphi = Z\varphi'$ for each $Z \in \mathcal{B}_q^{\mathcal{E}}(Y)$. ■

For each relational language $\mathcal{L}$, each $\mathcal{L}$-structure M and each subgroup H of $\mathrm{Aut}(M)$, we define the $\mathcal{L}$-structure M/H as follows: The elements of M/H are the classes xH for $x \in M$. For $R(u_1, ..., u_k) \in \mathcal{L}$ and $x_1, ..., x_k \in M/H$, we write $R(x_1, ..., x_k)$ if there exist some representatives $y_1, ..., y_k$ of $x_1, ..., x_k$ in M such that $R(y_1, ..., y_k)$.

For each $i \in \{1, ..., k\}$ and each representative y_i of x_i in M, there exist some representatives $y_1, ..., y_{i-1}, y_{i+1}, ..., y_k$ of $x_1, ..., x_{i-1}, x_{i+1}, ..., x_k$ in M such that $R(y_1, ..., y_k)$. The canonical surjection from M to M/H is a homomorphism. If H is normal in $\mathrm{Aut}(M)$, then any automorphism of M induces an automorphism of M/H.

In Proposition 2.8 below, we do not use the supplementary hypotheses on (E,δ) which were introduced for Theorem 2.7. For each Δ-tiling $\mathcal{T}$ and each subgroup H of G such that $T\sigma \in \mathcal{T} - \mathcal{B}_{2q}^{\mathcal{T}}(T)$ for each $T \in \mathcal{T}$ and each $\sigma \in H$, we denote by $\mathcal{T}/H$ the tiling of E/H induced by $\mathcal{T}$. The canonical surjection $\pi : \mathcal{T} \to \mathcal{T}/H$ is a $\mathcal{L}_\Delta$-homomorphism. Proposition 2.8 implies that, in Theorem 2.7, the $\mathcal{L}_\Delta$-structure M is isomorphic to a quotient of a Δ-tiling.

Proposition 2.8. Consider a Δ-tiling $\mathcal{T}$, a $\mathcal{L}_\Delta$-structure M and a surjective $\mathcal{L}_\Delta$-homomorphism $\varphi : \mathcal{T} \to M$ which induces a $\mathcal{L}_\Delta$-isomorphism from $\mathcal{B}_q^{\mathcal{T}}(T)$ to $B_q^M(T\varphi)$ for each $T \in \mathcal{T}$. Then φ induces a $\mathcal{L}_\Delta$-isomorphism from $\mathcal{T}/H$ to M, where $H = \{\sigma \in G \mid \mathcal{T}\sigma = \mathcal{T}$ and $T\sigma\varphi = T\varphi$ for each $T \in \mathcal{T}\}$.

Proof. First we show that φ induces a bijective homomorphism from $\mathcal{T}/H$ to M. For any $S,T \in \mathcal{T}$ such that $S\varphi = T\varphi$, as φ induces some $\mathcal{L}_\Delta$-isomorphisms from $\mathcal{B}_q^{\mathcal{T}}(S)$ and $\mathcal{B}_q^{\mathcal{T}}(T)$ to $B_q^M(S\varphi) = B_q^M(T\varphi)$, Theorem 1.3 implies that there exists a unique $\sigma_{S,T} \in G$ such that $S\sigma_{S,T} = T$ and $U\varphi = U\sigma_{S,T}\varphi$ for each $U \in \mathcal{B}_q^{\mathcal{T}}(S)$. For any $S,T \in \mathcal{T}$ such that $S\varphi = T\varphi$ and for each $U \in \mathcal{B}_1^{\mathcal{T}}(S)$, we have $\sigma_{S,T} = \sigma_{U,V}$ where $V = U\sigma_{S,T}$, since $\sigma_{S,T}$ and $\sigma_{U,V}$ coincide on $\mathcal{B}_{q-1}^{\mathcal{T}}(S)$. Consequently, for any $S,T \in \mathcal{T}$ such that $S\varphi = T\varphi$, we have $\mathcal{T}\sigma_{S,T} = \mathcal{T}$ and $U\varphi = U\sigma_{S,T}\varphi$ for each $U \in \mathcal{T}$.

It remains to be proved that, for each $R(w_1,...,w_k) \in \mathcal{L}_\Delta$ and any $T_1,...,T_k \in \mathcal{T}$ such that M satisfies $R(T_1\varphi,...,T_k\varphi)$, there exist $U_1 \in T_1H,...,U_k \in T_kH$ such that $\mathcal{T}$ satisfies $R(U_1,...,U_k)$. As φ induces a $\mathcal{L}_\Delta$-isomorphism from $\mathcal{B}_q^{\mathcal{T}}(T_1)$ to $B_q^M(T_1\varphi)$, there exist $U_2,...,U_k \in \mathcal{B}_q^{\mathcal{T}}(T_1)$ such that $R(T_1,U_2,...,U_k)$ and $U_2\varphi = T_2\varphi,...,U_k\varphi = T_k\varphi$. We have $U_2 \in T_2H,...,U_k \in T_kH$ according to the first part of the proof. ■

3. Periodicity, invariance through a translation.

In the present section, we consider a finite relational language $\mathcal{L}$. We generalize to uniformly locally finite $\mathcal{L}$-structures the notions of periodicity and invariance through a nontrivial translation, which are usually considered for tilings of the euclidean spaces $\mathbb{R}^n$. In particular, we obtain generalizations for tilings of noneuclidean spaces. The notions of mathematical logic used for Proposition 3.1 and Corollary 3.2 are defined, for instance, in [5].

Proposition 3.1. Consider a formula $\theta(u,v)$ in $\mathcal{L}$ and two elementarily equivalent $\mathcal{L}$-structures M,N with M connected locally finite. Suppose that there exist an element $x \in M$ such that $\{y \in M \mid \theta(x,y)\}$ is finite, and an automorphism g of N such that $\theta(y,yg)$ for each $y \in N$. Then there exists an automorphism f of M such that $\theta(y,yf)$ for each $y \in M$. Moreover, for each $r \in \mathbb{N}$, if $\{y \in N \mid yg = y\}$ contains no ball $B_N(z,r)$, then we can choose f in such a way that $\{y \in M \mid yf = y\}$ contains no ball $B_M(z,r)$.

Proof. Fix $x \in M$ such that $\{y \in M \mid \theta(x,y)\}$ is finite. For each $k \in \mathbb{N}$, as $B_M(x,k)$ is finite, the following property of a $\mathcal{L}$-structure P can be expressed by one sentence:
For each $y \in P$ such that $(B_P(y,k),y) \cong (B_M(x,k),x)$, there exist an element $z \in P$ and an isomorphism $f : (B_P(y,k),y) \to (B_P(z,k),z)$ such that $\theta(u,uf)$ for each $u \in B_P(y,k)$ (respectively $\theta(u,uf)$ for each $u \in B_P(y,k)$ and $\{v \in B_P(y,k) \mid vf = v\}$ contains no ball $B_P(u,r)$ for $u \in B_P(y,k-r)$).

Suppose that there exists an automorphism g of N such that $\theta(y,yg)$ for each $y \in N$ (respectively $\theta(y,yg)$ for each $y \in N$ and $\{y \in N \mid yg = y\}$ contains no ball $B_N(z,r)$). Then the sentence considered above is true in N, and therefore true in M.

For each $k \in \mathbb{N}$, consider the nonempty set A_k consisting of the pairs (y,f) with $y \in M$ and $f : (B_M(x,k),x) \to (B_M(y,k),y)$ isomorphism such that $\theta(u,uf)$ for each $u \in B_M(x,k)$ (respectively $\theta(u,uf)$ for each $u \in B_M(x,k)$ and $\{v \in B_M(x,k) \mid vf = v\}$ contains no ball $B_M(u,r)$ for $u \in B_M(x,k-r)$). Then each A_k is finite since M is locally finite and $\{y \in M \mid \theta(x,y)\}$ is finite. Moreover, for $0 \leq k \leq l$, each pair $(z,g) \in A_l$ gives by restriction a pair $(y,f) \in A_k$.

Consequently, by König's lemma, there exists a sequence $(y_k,f_k)_{k\in\mathbb{N}} \in \Pi_{k\in\mathbb{N}} A_k$ with f_k restriction of f_l for $0 \leq k \leq l$. The inductive limit of such a sequence gives an automorphism of M which satisfies the required properties. ■

By [7, Theorem 2.3], two locally finite $\mathcal{L}$-structures are elementarily equivalent if and only if they are locally isomorphic. Moreover, for any $r,s \in \mathbb{N}^*$, there exists a formula $\theta_{r,s}(u,v)$ which expresses the property $v \in B_N(u,r)$ in each $\mathcal{L}$-structure N such that $|B_N(x,r)| \leq s$ for each $x \in N$. Consequently, we have:

Corollary 3.2. Consider two locally isomorphic uniformly locally finite $\mathcal{L}$-structures M,N which M connected. Let $r \in \mathbb{N}^*$ and $s \in \mathbb{N}$. Suppose that there exists an automorphism g of N such that $yg \in B_N(y,r)$ for each $y \in N$, and such that $\{y \in N \mid yg = y\}$ contains no ball $B_N(z,s)$. Then, there exists an automorphism f of M such that $yf \in B_M(y,r)$ for each $y \in M$, and such that $\{y \in M \mid yf = y\}$ contains no ball $B_M(z,s)$.

Remark. In particular, for each $r \in \mathbb{N}^*$, if N has an automorphism g without fixed point such that $yg \in B_N(y,r)$ for each $y \in N$, then M has an automorphism f without fixed point such that $yf \in B_M(y,r)$ for each $y \in M$.

Now we consider the metric space (E,δ), the group G of isometries of E and the set Δ defined in Section 1. For each Δ-tiling $\mathcal{T}$ and each $\sigma \in G$ such

that $\mathcal{T}\sigma = \mathcal{T}$, we say that σ is a *translation of* $\mathcal{T}$ if there exists $r \in \mathbb{N}^*$ such that $T\sigma \in \mathcal{B}_r^{\mathcal{T}}(T)$ for each $T \in \mathcal{T}$. The set $\mathrm{Trans}(\mathcal{T})$ of all translations of $\mathcal{T}$ is a subgroup of G.

Any $\sigma \in G$ such that $\mathcal{T}\sigma = \mathcal{T}$ is a translation of $\mathcal{T}$ if and only if there exists $s \in \mathbb{N}^*$ such that $T\sigma \in B_{\mathcal{T}}(T, s)$ for each $T \in \mathcal{T}$. Moreover, for each $T \in \mathcal{T}$, there exists no $\sigma \in G - \{\mathrm{Id}\}$ such that $S\sigma = S$ for each $S \in B_{\mathcal{T}}(T, 1) = \mathcal{B}_{2q}^{\mathcal{T}}(T)$. Consequently, the result below follows from Corollary 3.2:

Corollary 3.3. For any locally isomorphic Δ-tilings $\mathcal{S}, \mathcal{T}$, we have $\mathrm{Trans}(\mathcal{S}) = \{\mathrm{Id}\}$ if and only if $\mathrm{Trans}(\mathcal{T}) = \{\mathrm{Id}\}$.

For each Δ-tiling $\mathcal{T}$ and each $\sigma \in \mathrm{Trans}(\mathcal{T})$, $\sup_{x \in E} \delta(x, x\sigma)$ is finite. Conversely, if (E, δ) is weakly homogeneous, then the second part of Proposition 1.2 implies that any $\sigma \in G$ such that $\mathcal{T}\sigma = \mathcal{T}$ belongs to $\mathrm{Trans}(\mathcal{T})$ if $\sup_{x \in E} \delta(x, x\sigma)$ is finite. For each $k \in \mathbb{N}^*$, if E is the set $\mathbb{R}^k$ equipped with a distance defined from a norm, then we have $\sigma \in \mathrm{Trans}(\mathcal{T})$ if and only if σ is a translation in the usual sense, since any surjective isometry of E is affine in that case by [2, Th. 14.1].

The following result generalizes well known properties of the translations in the euclidean spaces $\mathbb{R}^k$. Here, we use the conditions (CVXν) and (FIXν) introduced at the end of Section 1.

Theorem 3.4. Let $\mathcal{T}$ be a Δ-tiling. Then, for each $\sigma \in \mathrm{Trans}(\mathcal{T}) - \{\mathrm{Id}\}$, there exist $x, y \in E$ such that $\delta(x, x\sigma) = \inf_{z \in E} \delta(z, z\sigma)$ and $\delta(y, y\sigma) = \sup_{z \in E} \delta(z, z\sigma)$. Moreover, if (E, δ) is geodesic and satisfies (CVXν) and (FIXν) for each $\nu \in \mathbb{R}_{>0}$, then $\mathrm{Trans}(\mathcal{T})$ is torsion-free abelian and each $\sigma \in \mathrm{Trans}(\mathcal{T}) - \{\mathrm{Id}\}$ has no fixed point.

Proof. We fix $\sigma \in \mathrm{Trans}(\mathcal{T}) - \{\mathrm{Id}\}$, we write $\alpha = \inf_{z \in E} \delta(z, z\sigma)$ and we show that there exists $x \in E$ such that $\delta(x, x\sigma) = \alpha$. It can be proved in a similar way that there exists $y \in E$ such that $\delta(y, y\sigma) = \sup_{z \in E} \delta(z, z\sigma)$.

We consider $r \in \mathbb{N}^*$ such that $T\sigma \in \mathcal{B}_r^{\mathcal{T}}(T)$ for each $T \in \mathcal{T}$. For each $T \in \mathcal{T}$, σ induces a bijection $\sigma_T : \mathcal{B}_q^{\mathcal{T}}(T) \to \mathcal{B}_q^{\mathcal{T}}(T\sigma)$. The triples $(\mathcal{B}_{q+r}^{\mathcal{T}}(T), T, \sigma_T)$ for $T \in \mathcal{T}$ fall in finitely many isomorphism classes. Consequently, there exist two sequences $(x_k)_{k \in \mathbb{N}} \subset E$ and $(T_k)_{k \in \mathbb{N}} \subset \mathcal{T}$ with $x_k \in T_k$ for each $k \in \mathbb{N}$ such that $\lim \delta(x_k, x_k\sigma) = \alpha$ and such that all the triples $(\mathcal{B}_{q+r}^{\mathcal{T}}(T_k), T_k, \sigma_{T_k})$ are isomorphic.

For each $k \in \mathbb{N}$, we consider $\tau_k \in G$ which induces an isomorphism from $(\mathcal{B}_{q+r}^{\mathcal{T}}(T_0), T_0, \sigma_{T_0})$ to $(\mathcal{B}_{q+r}^{\mathcal{T}}(T_k), T_k, \sigma_{T_k})$. We have $\sigma = \tau_k \sigma \tau_k^{-1}$ for each $k \in \mathbb{N}$ since $T\sigma = T\sigma_{T_0} = T\tau_k \sigma_{T_k} \tau_k^{-1} = T\tau_k \sigma \tau_k^{-1}$ for each $T \in \mathcal{B}_q^{\mathcal{T}}(T_0)$. Consequently, the elements $y_k = x_k \tau_k^{-1} \in T_0$ satisfy $\delta(y_k, y_k\sigma) = \delta(x_k, x_k\sigma)$,

and therefore $\lim \delta(y_k, y_k\sigma) = \alpha$. As T_0 is compact, it follows that there exists $x \in T_0$ such that $\delta(x, x\sigma) = \alpha$.

Now suppose that (E, δ) is geodesic and satisfies (CVXν) and (FIXν) for each $\nu \in \mathbb{R}_{>0}$. Consider $\sigma \in \mathrm{Trans}(\mathcal{T})$ and $s \in \mathbb{N} - \{0, 1\}$ such that $\sigma^s = \mathrm{Id}$. Then, for each $x \in E$, we have $A_x\sigma = A_x$ for $A_x = \{x, x\sigma, ..., x\sigma^{s-1}\}$; by Lemma 1.9, there exists a unique $w_x \in E$ such that $\delta(w_x, z) \leq \mathrm{Rad}(A_x)$ for each $z \in A_x$; it follows $w_x\sigma = w_x$. The properties (FIXν) imply $\sigma = \mathrm{Id}$ since $\delta(x, w_x) \leq \mathrm{Rad}(A_x) \leq (s/2) \sup_{z\in E} \delta(z, z\sigma)$ for each $x \in E$.

Now consider any $\sigma \in \mathrm{Trans}(\mathcal{T})$ with a fixed point x. Then, for each $y \in E$, there exists $s \in \mathbb{N}^*$ such that $y\sigma^s = y$, since $\mathrm{Trans}(\mathcal{T})$ is discrete by Proposition 1.8 and the elements $y\sigma^k$ for $k \in \mathbb{Z}$ all belong to $\beta(x, \delta(x, y))$. Consider $\zeta, \eta \in \mathbb{R}_{>0}$ such that no $\tau \in G - \{\mathrm{Id}\}$ fixes the points of a set $A \subset E$ with $\beta(y, \eta) \subset \cup_{z\in A}\beta(z, \zeta)$ for some $y \in E$, and choose a finite set A with that property. Then there exists $s \in \mathbb{N}^*$ such that $z\sigma^s = z$ for each $z \in A$, which implies $\sigma^s = \mathrm{Id}$ and $\sigma = \mathrm{Id}$.

It remains to be proved that any $\sigma, \tau \in \mathrm{Trans}(\mathcal{T})$ commute. It suffices to show that σ commutes with τ^r for some $r \in \mathbb{Z}^*$, since $(\sigma^{-1}\tau\sigma)^r = \sigma^{-1}\tau^r\sigma = \mathrm{Id}$ implies $\sigma^{-1}\tau\sigma = \mathrm{Id}$.

We fix $x \in E$ and we consider $\alpha \in \mathbb{R}_{>0}$ such that $\delta(y, y\sigma) \leq \alpha$ for each $y \in E$. For each $r \in \mathbb{Z}$, we have $\delta(x, x\sigma^{-1}\tau^{-r}\sigma\tau^r) \leq 2\alpha$ since $\delta(x, x\sigma^{-1}) \leq \alpha$ and $\delta(x\sigma^{-1}, x\sigma^{-1}\tau^{-r}\sigma\tau^r) = \delta(x\sigma^{-1}\tau^{-r}, x\sigma^{-1}\tau^{-r}\sigma) \leq \alpha$.

As $\mathrm{Trans}(\mathcal{T})$ is discrete, it follows that there exist $r \neq s$ in $\mathbb{Z}$ such that $x\sigma^{-1}\tau^{-r}\sigma\tau^r = x\sigma^{-1}\tau^{-s}\sigma\tau^s$. We have $\sigma^{-1}\tau^{-r}\sigma\tau^r = \sigma^{-1}\tau^{-s}\sigma\tau^s$ since the elements of $\mathrm{Trans}(\mathcal{T})$ have no fixed point. It follows $\tau^{-r}\sigma\tau^r = \tau^{-s}\sigma\tau^s$ and $\tau^{s-r}\sigma = \sigma\tau^{s-r}$. ■

Now we generalize the notion of periodicity to connected $\mathcal{L}$-structures. We say that such a structure M is *periodic* if it contains a finite set A such that $M = \cup_{\sigma\in\mathrm{Aut}(M)} A\sigma$.

Proposition 3.5. Let M, N be locally isomorphic locally finite connected $\mathcal{L}$-structures. If N is periodic then M is isomorphic to N.

Proof. We fix $w \in M$. As N is periodic and connected, there exist $z \in N$ and $r \in \mathbb{N}$ such that $N = \cup_{\sigma\in\mathrm{Aut}(N)} B_N(z, r)\sigma$. For each $s \in \mathbb{N}$, as M and N are locally isomorphic, there exists $x_s \in N$ such that $(B_M(w, s), w) \cong (B_N(x_s, s), x_s)$, and therefore $y_s \in B_N(z, r)$ such that $(B_M(w, s), w) \cong (B_N(y_s, s), y_s)$. We consider the nonempty set A_s consisting of the isomorphisms $\theta : (B_M(w, s), w) \to (B_N(y, s), y)$ with $y \in B_N(z, r)$.

The sets A_s are finite since N is locally finite. Moreover, for any $s \leq t$, the restriction of each $\theta \in A_t$ to $(B_M(w, s), w)$ belongs to A_s. Consequently, by König's Lemma, there exists a strictly increasing sequence $(\theta_s)_{s\in\mathbb{N}}$ with

$\theta_s \in A_s$ for each $s \in \mathbb{N}$. The inductive limit is an isomorphism $\theta : (M, w) \to (N, y)$ with $y \in B_N(z, r)$. ■

We call a *period* of a $\mathcal{L}$-structure M any set $A \subset M$ such that M is the disjoint union of the sets $A\sigma$ for $\sigma \in \mathrm{Aut}(M)$. If M is periodic and if it has some periods, then all of them have the same finite number of elements. We call it the *periodicity rank* of M.

For each $\mathcal{L}$-structure M, we say that $A \subset M$ is *weakly connected* if, for each subset B with $\varnothing \subsetneq B \subsetneq A$, there exist $k \in \mathbb{N} - \{0, 1\}$ and $R(u_1, ..., u_k) \in \mathcal{L}$ which is satisfied by some $(x_1, ..., x_k) \in M^k$ with $\{x_1, ..., x_k\} \cap B \neq \varnothing$ and $\{x_1, ..., x_k\} \cap (A - B) \neq \varnothing$.

Proposition 3.6. Any connected $\mathcal{L}$-structure M has a weakly connected period if the nontrivial automorphisms of M have no fixed point.

Proof. We show that $M = \cup_{\sigma \in \mathrm{Aut}(M)} A\sigma$ for each $A \subset M$ which is maximal for the conjunction of the two properties: A weakly connected and $A \cap A\sigma = \varnothing$ for each $\sigma \in \mathrm{Aut}(M) - \{\mathrm{Id}\}$.

Otherwise, as M is connected, there exist $k \in \mathbb{N} - \{0, 1\}$, $(x_1, ..., x_k) \in M^k$ which satisfies some $R(u_1, ..., u_k) \in \mathcal{L}$ and $1 \leq i, j \leq k$ such that $x_i \in \cup_{\sigma \in \mathrm{Aut}(M)} A\sigma$ and $x_j \in M - \cup_{\sigma \in \mathrm{Aut}(M)} A\sigma$. We consider $\tau \in \mathrm{Aut}(M)$ such that $x_i\tau \in A$. Then $B = A \cup \{x_j\tau\}$ is weakly connected since $(x_1\tau, ..., x_k\tau)$ satisfies R. As $x_j\tau$ is not a fixed point of a nontrivial automorphism of M and does not belong to $\cup_{\sigma \in \mathrm{Aut}(M)} A\sigma$, we have $B\sigma \cap B = \varnothing$ for each $\sigma \in \mathrm{Aut}(M) - \{\mathrm{Id}\}$, contrary to the maximality of A. ■

Now we introduce some supplementary conditions which will be used for the investigation of periodic $\mathcal{L}$-structures.

We say that a $\mathcal{L}$-structure M is *equational* (cf. [7, Section 4]) if $R(x_1, ..., x_k)$, $R(y_1, ..., y_k)$ and $x_i = y_i$ imply $x_j = y_j$ for each $k \in \mathbb{N}^*$, each $R(u_1, ..., u_k) \in \mathcal{L}$, any $(x_1, ..., x_k), (y_1, ..., y_k) \in M^k$ and any $i, j \in \{1, ..., k\}$.

Any equational $\mathcal{L}$-structure is uniformly locally finite. If it is connected, then its nontrivial automorphisms have no fixed point.

For each equational $\mathcal{L}$-structure M, each $k \in \mathbb{N} - \{0, 1\}$, each $R(u_1, ..., u_k) \in \mathcal{L}$, any $i, j \in \{1, ..., k\}$ such that $i \neq j$ and any $x, y \in M$, we write $x(R, i, j) = y$ if there exists $(z_1, ..., z_k) \in M^k$ which satisfies R with $z_i = x$ and $z_j = y$.

For each $n \in \mathbb{N}$, each word $w = (R_1, i_1, j_1)...(R_n, i_n, j_n)$ and any $x, y \in M$, we write $xw = y$ if there exist $z_0, ..., z_n \in M$ such that $z_0 = x$, $z_n = y$ and $z_{m-1}(R_m, i_m, j_m) = z_m$ for $1 \leq m \leq n$. We denote by $\Omega_{\mathcal{L}}$ the set of all such words.

For any equational $\mathcal{L}$-structures M, N, each $x \in M$ and each $r \in \mathbb{N}^*$, we call a *partial isomorphism* from $B_M(x, r)$ to N any injective map ρ such that, for each $k \in \mathbb{N}^*$, each $R(u_1, ..., u_k) \in \mathcal{L}$, any $i, j \in \{1, ..., k\}$ and any $y, z \in$

$B_M(x,r)$, $y(R,i,j)$ exists if and only if $y\rho(R,i,j)$ exists and $y(R,i,j)=z$ if and only if $y\rho(R,i,j)=z\rho$.

If ρ is a partial isomorphism from $B_M(x,r)$ to N, then ρ is an isomorphism from $(B_M(x,r),x)$ to $(B_N(x\rho,r),x\rho)$ and ρ^{-1} is a partial isomorphism from $B_N(x\rho,r)$ to M. Any isomorphism $\sigma:(B_M(x,r+1),x)\to(B_N(x\sigma,r+1),x\sigma)$ gives by restriction a partial isomorphism from $B_M(x,r)$ to N.

We say that an equational $\mathcal{L}$-structure M is *commutative* if, for each $x\in M$ and each $r\in\mathbb{N}^*$, each partial isomorphism $\rho:B_M(x,r)\to M$ satisfies $y\rho\sigma=y\sigma\rho$ for each automorphism σ of M and each $y\in B_M(x,r)$ such that $y\sigma\in B_M(x,r)$.

We say that M is *strongly commutative* if we have $xvw=xwv$ for each $x\in M$ and any $v,w\in\Omega_{\mathcal{L}}$ such that xvw and xwv exist. Any strongly commutative connected equational $\mathcal{L}$-structure is commutative since, for x,y,ρ,σ defined as above, there exist $v,w\in\Omega_{\mathcal{L}}$ such that the equalities $y\rho=yv$ and $y\sigma=yw$ are respectively true in M and in $B_M(x,r)$, which implies $y\rho\sigma=yvw$ and $y\sigma\rho=ywv$.

We say that a class $\mathcal{E}$ of equational $\mathcal{L}$-structures is *strongly regular* (cf. [7, Section 4]) if $xw=x$ and $yw=y$ are equivalent for any $M,N\in\mathcal{E}$, each $x\in M$, each $y\in N$ and each $w\in\Omega_{\mathcal{L}}$ such that xw and yw exist.

Equationality, strong commutativity and strong regularity are local properties. They are preserved by local isomorphism.

Proposition 3.7. Let M be a connected equational commutative $\mathcal{L}$-structure. If M is periodic of rank $r\in\mathbb{N}^*$, then, for each $x\in M$ and each integer $s\geq r$, each partial isomorphism from $B_M(x,s)$ to M can be extended into a unique automorphism of M.

Proof. The extension is necessarily unique since the nontrivial automorphisms of an equational $\mathcal{L}$-structure have no fixed point. It remains to be proved that it exists.

By Proposition 3.6, M has a weakly connected period A with $|A|=r$. Replacing A by its image through an appropriate automorphism of M, we reduce the proof to the case $x\in A$. Then A is contained in $B_M(x,r-1)$.

For each partial isomorphism $\rho:B_M(x,s)\to M$, we define a map $\overline{\rho}:M\to M$ as follows: for each $y\in M$, as the nontrivial automorphisms of M have no fixed point, there exists a unique $\sigma\in\mathrm{Aut}(M)$ such that $y\sigma\in A$; we write $y\overline{\rho}=y\sigma\rho\sigma^{-1}$. For each $y\in B_M(x,s)$, we have $y\overline{\rho}=y\rho$ because the commutativity of M implies $y\rho\sigma=y\sigma\rho$. We observe that $\overline{\rho}^{-1}$ is defined in the same way from the partial isomorphism $\rho^{-1}:B_M(x\rho,s)\to M$ and the weakly connected period $A\rho$. Consequently, $\overline{\rho}$ is bijective and, by reason of symmetry, it suffices to show that $\overline{\rho}$ is a homomorphism.

For each $k\in\mathbb{N}^*$, each $R(u_1,...,u_k)\in\mathcal{L}$ and each $(x_1,...,x_k)\in M^k$ which satisfies R, we consider $\sigma_1,...,\sigma_k\in\mathrm{Aut}(M)$ such that $x_1\sigma_1,...,x_k\sigma_k$ belong

to A. Then $x_1\sigma_1, ..., x_k\sigma_k$ belong to $B_M(x, r-1)$ and $x_2\sigma_1, ..., x_k\sigma_1$ belong to $B_M(x, r)$ since $(x_1\sigma_1, x_2\sigma_1, ..., x_k\sigma_1)$ satisfies R.

We have $x_1\overline{\rho} = x_1\sigma_1\rho\sigma_1^{-1}$. For $2 \leq i \leq k$, we have $x_i\overline{\rho} = x_i\sigma_i\rho\sigma_i^{-1} = (x_i\sigma_1)(\sigma_1^{-1}\sigma_i)\rho(\sigma_1^{-1}\sigma_i)^{-1}\sigma_1^{-1}$. As $x_i\sigma_1$ and $(x_i\sigma_1)(\sigma_1^{-1}\sigma_i) = x_i\sigma_i$ belong to $B_M(x, r)$, the commutativity of M implies
$(x_i\sigma_1)(\sigma_1^{-1}\sigma_i)\rho = (x_i\sigma_1)\rho(\sigma_1^{-1}\sigma_i)$, and therefore
$x_i\overline{\rho} = (x_i\sigma_1)(\sigma_1^{-1}\sigma_i)\rho(\sigma_1^{-1}\sigma_i)^{-1}\sigma_1^{-1} = (x_i\sigma_1)\rho(\sigma_1^{-1}\sigma_i)(\sigma_1^{-1}\sigma_i)^{-1}\sigma_1^{-1} = x_i\sigma_1\rho\sigma_1^{-1}$.

Moreover, $x_1\sigma_1\rho, x_2\sigma_1\rho, ..., x_k\sigma_1\rho$ are all defined since $x_1\sigma_1, x_2\sigma_1, ..., x_k\sigma_1$ belong to $B_M(x, r)$. Consequently, $(x_1\sigma_1\rho, x_2\sigma_1\rho, ..., x_k\sigma_1\rho)$ satisfies R like $(x_1\sigma_1, x_2\sigma_1, ..., x_k\sigma_1)$ and
$(x_1\overline{\rho}, x_2\overline{\rho}, ..., x_k\overline{\rho}) = (x_1\sigma_1\rho\sigma_1^{-1}, x_2\sigma_1\rho\sigma_1^{-1}, ..., x_k\sigma_1\rho\sigma_1^{-1})$ also satisfies R. ■

Lemma 3.8. Let M, N be equational $\mathcal{L}$-structures with N connected, commutative and periodic of rank $r \in \mathbb{N}^*$. For each $x \in M$ and each integer $s \geq r$, if there exists a partial isomorphism from $B_M(x, s+1)$ to N, then each partial isomorphism from $B_M(x, s)$ to N can be extended into a partial isomorphism from $B_M(x, s+1)$ to N.

Proof. Let us consider a partial isomorphism $\rho : B_M(x, s) \to N$ and a partial isomorphism $\sigma : B_M(x, s+1) \to N$. Then $\rho^{-1}\sigma$ is a partial isomorphism from $B_N(x\rho, s)$ to N which can be extended into a unique automorphism θ of N by Proposition 3.7, and $\sigma\theta^{-1}$ is a partial isomorphism from $B_M(x, s+1)$ to N which extends ρ. ■

Lemma 3.9. Let M, N be equational $\mathcal{L}$-structures with $\{M, N\}$ strongly regular. Then, for each $x \in M$ and each $r \in \mathbb{N}^*$, each partial isomorphism $\rho : B_M(x, r) \to N$ can be extended into a partial isomorphism from $B_M(x, r+1)$ to N if, for each $y \in B_M(x, r)$, there exists a partial isomorphism $\rho_y : B_M(y, 1) \to N$ such that $y\rho_y = y\rho$.

Proof. We define a map $\sigma : B_M(x, r+1) \to N$ as follows: for each $z \in B_M(x, r+1)$, we consider $y \in B_M(x, r)$ such that $z \in B_M(y, 1)$ and we write $z\sigma = z\rho_y$.

For any $z_1, z_2 \in B_M(x, r+1)$ and any $y_1, y_2 \in B_M(x, r)$ such that $z_1 \in B_M(y_1, 1)$ and $z_2 \in B_M(y_2, 1)$, there exist $v, (R_1, i_1, j_1), (R_2, i_2, j_2) \in \Omega_{\mathcal{L}}$ such that the equalities $y_1 v = y_2$, $y_1(R_1, i_1, j_1) = z_1$ and $y_2(R_2, i_2, j_2) = z_2$ are respectively true in the connected $\mathcal{L}$-structures $B_M(x, r)$, $B_M(y_1, 1)$ and $B_M(y_2, 1)$. We have $z_2 = z_1 w$ for $w = (R_1, j_1, i_1)v(R_2, i_2, j_2)$. We also have $z_2\rho_{y_2} = z_1\rho_{y_1} w$ since the equalities $z_1\rho_{y_1}(R_1, j_1, i_1) = y_1\rho_{y_1} = y_1\rho$, $y_1\rho v = y_2\rho = y_2\rho_{y_2}$ and $y_2\rho_{y_2}(R_2, i_2, j_2) = z_2\rho_{y_2}$ are respectively true in $B_N(y_1\rho_{y_1}, 1)$, $B_N(x\rho, r)$ and $B_N(y_2\rho_{y_2}, 1)$.

By strong regularity, the equalities $z_2 = z_1 w$ and $z_2\rho_{y_2} = z_1\rho_{y_1} w$ imply that $z_1\rho_{y_1} = z_2\rho_{y_2}$ if and only if $z_1 = z_2$. Consequently, σ is injective and the

definition of $z\sigma$ given above does not depend on the choice of the element $y \in B_M(x,r)$ such that $z \in B_M(y,1)$. It follows that $z\sigma = z\rho$ for each $z \in B_M(x,r)$ and $z\sigma = z\rho_y$ for each $y \in B_M(x,r)$ and each $z \in B_M(y,1)$. For each $z \in B_M(x,r+1)$ and each $(R,i,j) \in \Omega_{\mathcal{L}}$, the element $z\sigma(R,i,j)$ exists if and only if $z(R,i,j)$ exists, because, for each $y \in B_M(x,r)$, $z\rho_y(R,i,j)$ exists if and only if $z(R,i,j)$ exists.

Now let us consider again y_1, y_2, z_1, z_2, w as above. For each(R,i,j) such that $z_1(R,i,j)$ exists, or equivalently such that $z_1\sigma(R,i,j)$ exists, we have $z_1(R,i,j)w' = z_2$ and $z_1\sigma(R,i,j)w' = z_2\sigma$ for $w' = (R,j,i)w$. Strong regularity implies that $z_1(R,i,j) = z_2$ if and only if $z_1\sigma(R,i,j) = z_2\sigma$. ■

Theorem 3.10. Let M, N be connected periodic $\mathcal{L}$-structures of ranks $r, s \geq 1$ with N commutative and $\{M, N\}$ strongly regular. Then, for each $x \in M$, each partial isomorphism from $B_M(x, r+s)$ to N can be extended into an isomorphism from M to N.

Proof. Otherwise, there exist an integer $t \geq r+s$ and a partial isomorphism $\rho : B_M(x,t) \to N$ which cannot be extended into a partial isomorphism from $B_M(x,t+1)$ to N. By lemma 3.9, there exists $y \in B_M(x,t) - B_M(x,t-1)$ such that no partial isomorphism $\sigma : B_M(y,1) \to N$ satisfies $y\sigma = y\rho$.

For each $k \in \{0, ..., r\}$, we consider $z_k \in B_M(x, t-s-k) - B_M(x, t-s-k-1)$ such that $y \in B_M(z_k, s+k)$. The restriction ρ_k of ρ to $B_M(z_k, s+k)$ cannot be extended into a partial isomorphism from $B_M(z_k, s+k+1)$ to N.

As M is periodic of rank r, there exist two integers $0 \leq k_1 < k_2 \leq r$ and an isomorphism $\sigma : (M, z_{k_1}) \to (M, z_{k_2})$. Then $\sigma\rho_{k_2}$ is a partial isomorphism from $B_M(z_{k_1}, s+k_2)$ to N. By Lemma 3.8, it follows that ρ_{k_1} can be extended into a partial isomorphism from $B_M(z_{k_1}, s+k_2)$ to N, whence a contradiction. ■

Theorem 3.11. For each set Σ of local rules for $\mathcal{L}$, if the connected $\mathcal{L}$-structures satisfying Σ are periodic, equational and commutative, and if their class $\mathcal{E}$ is strongly regular, then $\mathcal{E}$ is a finite union of isomorphism classes.

Proof. Otherwise, it follows from Theorem 3.10 that there exists a sequence $(N_i)_{i\in\mathbb{N}}$ of periodic $\mathcal{L}$-structures with strictly increasing periodicity ranks which satisfy Σ. For each $q \in \mathbb{N}^*$, the pairs $(B_{N_i}(y,q), y)$ for $i \in \mathbb{N}$ and $y \in N_i$ fall in finitely many isomorphism classes. König's lemma applied to the isomorphism classes of pairs $(B_{N_i}(y,q), y)$ for $q \in \mathbb{N}^*$, $i \geq q$ and $y \in N_i$ implies that there exist two strictly increasing sequences of integers $(k(i))_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$, and a sequence $(y_i)_{i\in\mathbb{N}} \in \times_{i\in\mathbb{N}} N_{k(i)}$ such that $(B_{N_{k(i)}}(y_i, q_i), y_i) \cong (B_{N_{k(j)}}(y_j, q_i), y_j)$ for $i \leq j$.

Consequently, there exist a sequence $(M_i)_{i\in\mathbb{N}}$ of periodic $\mathcal{L}$-structures with strictly increasing periodicity ranks which satisfy Σ and a sequence $(x_i)_{i\in\mathbb{N}} \in$

$\times_{i\in\mathbb{N}} M_i$ such that $(B_{M_i}(x_i,i),x_i) \cong (B_{M_j}(x_j,i),x_j)$ for $i \leq j$. The inductive limit of the pairs $(B_{M_i}(x_i,i),x_i)$ for these isomorphisms, which are unique and compatible because of equationality, is a pair (M,x) with M satisfying Σ and $x \in M$. We denote by r the periodicity rank of M and, for each $i \in \mathbb{N}$, r_i the periodicity rank of M_i.

For each $i \geq r$, we have $r_i > r$. Consequently, M_i is not isomorphic to M. As $(B_{M_i}(x_i,i),x_i)$ and $(B_M(x,i),x)$ are isomorphic, there exist an integer $s_i \geq i-1$ and a partial isomorphism $\rho_i : B_{M_i}(x_i,s_i) \to M$ which cannot be extended into a partial isomorphism from $B_{M_i}(x_i,s_i+1)$ to M. By Lemma 3.9, there exists $y_i \in B_{M_i}(x_i,s_i) - B_{M_i}(x_i,s_i-1)$ such that no partial isomorphism $\sigma : B_{M_i}(y_i,1) \to M$ satisfies $y_i\sigma = y_i\rho_i$.

For any integers $i \geq r$ and $1 \leq s \leq s_i$, we consider $z_{i,s} \in B_{M_i}(x_i,s_i-s)$ such that $y_i \in B_{M_i}(z_{i,s},s)$. Then $B_{M_i}(y_i,2s)$ contains $B_{M_i}(z_{i,s},s)$.

It follows from König's Lemma applied to the pairs $(B_{M_i}(y_i,2s+1),y_i)$ for $i \geq r$ and $1 \leq s \leq s_i$ that there exist two strictly increasing sequences of integers $(k(i))_{i\in\mathbb{N}}$ and $(t_i)_{i\in\mathbb{N}}$, with $k(i) \geq r$ and $t_i \leq s_{k(i)}$ for each $i \in \mathbb{N}$, such that $(B_{M_{k(i)}}(y_{k(i)},2t_i+1),y_{k(i)}) \cong (B_{M_{k(j)}}(y_{k(j)},2t_i+1),y_{k(j)})$ for $i \leq j$. We denote by (N,y) the inductive limit of the pairs $(B_{M_{k(i)}}(y_{k(i)},2t_i+1),y_{k(i)})$ relative to these isomorphisms, which are unique and compatible because of equationality. The $\mathcal{L}$-structure N is periodic because it satisfies Σ. We denote by t the periodicity rank of N.

For each $i \in \mathbb{N}$, we have $y \in B_N(z_i,t_i)$ where z_i is the image of $z_{k(i),t_i}$ in N. Moreover $\rho_{k(i)}$ induces a partial isomorphism $\sigma_i : B_N(z_i,t_i) \to M$ such that no partial isomorphism $\tau : B_N(y,1) \to M$ satisfies $y\tau = y\sigma_i$. For $i \geq r+t-1$, this property contradicts Theorem 3.10 since we have $t_i \geq r+t$. ■

Now we consider again the metric space (E,δ), the group G of isometries of E and the set Δ defined in Section 1. We say that a Δ-tiling $\mathcal{T}$ is *periodic* if there exists a finite subset $\mathcal{E}$ of $\mathcal{T}$ such that $\mathcal{T}$ is the union of the subsets $\mathcal{E}\sigma$ for $\sigma \in G$ such that $\mathcal{T}\sigma = \mathcal{T}$. By Theorem 1.3, this property is true if and only if the $\mathcal{L}_\Delta$-structure associated to $\mathcal{T}$ is periodic.

For each $k \in \mathbb{N}^*$, if (E,δ) is the space $\mathbb{R}^k$ equipped with a distance defined from a norm and if G consists of the translations of E, then our notion of periodicity coincides with the classical one. Consequently, the following result generalizes [1, Th. 3.8] which was proved for tilings of $\mathbb{R}^2$ by square tiles:

Theorem 3.12. If (E,δ) is geodesic and satisfies (CVXλ) where λ is the maximum of the radii of the tiles, if G is commutative and if the elements of $G - \{\mathrm{Id}\}$ have no fixed point, then each class of periodic Δ-tilings defined by local rules is a finite union of isomorphism classes.

Lemma 3.12.1. The $\mathcal{L}_\Delta$-structures associated to Δ-tilings are equational, strongly commutative, and form a strongly regular class.

Proof of Lemma 3.12.1. If some $\sigma \in G$ stabilizes a tile, then $\sigma = \mathrm{Id}$ since σ has a fixed point by Lemma 1.9. Consequently, Theorem 1.3 implies that the $\mathcal{L}_\Delta$-structures associated to Δ-tilings are equational. Moreover, for any Δ-tilings $\mathcal{S}, \mathcal{T}$, each $(R, i, j) \in \Omega_{\mathcal{L}}$, each $S \in \mathcal{S}$ and each $T \in \mathcal{T}$, if $S(R, i, j)$ and $T(R, i, j)$ exist, then there exists a unique $\sigma \in G$ such that $S\sigma = T$ and it satisfies $S(R, i, j)\sigma = T(R, i, j)$. By induction, it follows that, for each $w \in \Omega_{\mathcal{L}}$, each $S \in \mathcal{S}$ and each $T \in \mathcal{T}$, if Sw and Tw exist, then there exists a unique $\sigma \in G$ such that $S\sigma = T$ and $Sw\sigma = Tw$. In particular, we have $Sw = S$ if and only if $Tw = T$.

For each Δ-tiling $\mathcal{T}$, each $T \in \mathcal{T}$ and any $v, w \in \Omega_{\mathcal{L}}$ such that Tvw and Twv exist, consider $\sigma, \tau \in G$ such that $Tv = T\sigma$, $Twv = Tw\sigma$, $Tw = T\tau$ and $Tvw = Tv\tau$. Then we have $Tvw = T\sigma\tau = T\tau\sigma = Twv$ since $\sigma\tau = \tau\sigma$. ■

Proof of Theorem 3.12. By Lemma 3.12.1, the $\mathcal{L}_\Delta$-structures associated to Δ-tilings are equational, commutative, and form a strongly regular class. If a class $\mathcal{C}$ of Δ-tilings is defined by local rules, then, by Theorem 2.1, the same property is true for the class K consisting of the associated $\mathcal{L}_\Delta$-structures. The structures in K are periodic if the tilings in $\mathcal{C}$ are periodic. Then Theorem 3.11 implies that K, and therefore $\mathcal{C}$, is a finite union of isomorphism classes. ■

4. Local isomorphism, rigidness and aperiodicity.

We say that a tiling or a relational structure is *rigid* if it has no nontrivial automorphism. In the present section, we are interested in characterizing tilings, and more generally relational structures, which are locally isomorphic to rigid ones. Theorem 4.1 (respectively Corollary 4.2) gives a characterization for relational structures (respectively tilings) which are uniformly locally finite and satisfy the local isomorphism property. Corollary 4.3 gives a simpler characterization concerning tilings of the euclidean spaces $\mathbb{R}^n$, where isomorphism is defined modulo a group of isometries.

In [7], we considered tilings of the euclidean spaces $\mathbb{R}^n$, and isomorphism was defined up to translation. In that case, [7, Proposition 2.4] implies that the relational structure M associated to a tiling is rigid if and only if the tiling is not invariant through any nontrivial translation. If a connected structure N is locally isomorphic to M, then N is associated to another tiling by [7, Corollary 5.4]. Moreover, according to [7, Proposition 5.1], the tilings associated to M and N are invariant through the same translations of $\mathbb{R}^n$. It follows that N is rigid if and only if M is rigid.

Examples 1, 2, 3, which are given after Corollary 4.3, imply that the last property is no longer true if we consider isomorphism modulo an arbitrary group of isometries of an euclidean space $\mathbb{R}^n$, or tilings of a noneuclidean space. Similarly, Example 4 illustrates Theorem 4.1 for relational structures

which are not associated to tilings. Examples 5 and 6 are given in order to show the importance of each hypothesis in Theorem 4.1.

Theorem 4.1. Let $\mathcal{L}$ be a finite relational language and let M be a uniformly locally finite $\mathcal{L}$-structure which satisfies the local isomorphism property. Then M is locally isomorphic to a connected rigid $\mathcal{L}$-structure if and only if, for each $r \in \mathbb{N}^*$, there exists $x \in M$ such that $(M, y) \cong (M, z)$ implies $y = z$ for $y, z \in B_M(x, r)$.

Proof. By König's lemma, the following properties are equivalent for locally finite $\mathcal{L}$-structures:
(P) There exist $r \in \mathbb{N}^*$ and, for each $x \in N$, $y \neq z$ in $B_N(x, r)$ such that $(N, y) \cong (N, z)$.
(Q) There exist $r \in \mathbb{N}^*$ and, for each $x \in N$ and each $s \in \mathbb{N}$, $y \neq z$ in $B_N(x, r)$ such that $(B_N(y, s), y) \cong (B_N(z, s), z)$.

Any $\mathcal{L}$-structure N which is locally isomorphic to M is uniformly locally finite like M. If M satisfies (P), and therefore satisfies (Q), then N satisfies (Q) since it locally isomorphic to M, and therefore satisfies (P). Consequently, N is not rigid.

Now, let us suppose that M does not satisfy (P). First we show that there exist a sequence $(x_n)_{n\in\mathbb{N}}$ in M and two strictly increasing sequences $(r_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that, for each $n \in \mathbb{N}$, $(B_M(x_n, r_n + s_n), x_n) \cong (B_M(x_{n+1}, r_n + s_n), x_{n+1})$ and $B_M(x_n, r_n)$ contains no elements $y \neq z$ with $(B_M(y, s_n), y) \cong (B_M(z, s_n), z)$. We write $r_0 = s_0 = 0$ and we take for x_0 any element of M.

For each $n \in \mathbb{N}$, supposing that x_n, r_n, s_n are already defined, we define $x_{n+1}, r_{n+1}, s_{n+1}$ as follows: As M satisfies the local isomorphism property, there exists $r \in \mathbb{N}$ such that each $B_M(x, r)$ contains some y with $(B_M(y, r_n + s_n), y) \cong (B_M(x_n, r_n + s_n), x_n)$; we take $r > r_n$. As M does not satisfy (Q), there exist $x \in M$ and $s \in \mathbb{N}^*$ such that $B_M(x, 2r)$ contains no elements $y \neq z$ with $(B_M(y, s), y) \cong (B_M(z, s), z)$. We take for x_{n+1} any $u \in B_M(x, r)$ such that $(B_M(u, r_n + s_n), u) \cong (B_M(x_n, r_n + s_n), x_n)$. Then $B_M(x_{n+1}, r)$ contains no elements $y \neq z$ with $(B_M(y, s), y) \cong (B_M(z, s), z)$. We take $r_{n+1} = r$ and $s_{n+1} = \sup(s_n + 1, s)$.

We consider the inductive limit (N, x) of the pairs $(B_M(x_n, r_n + s_n), x_n)$ relative to some isomorphisms

$$\theta_n : (B_M(x_n, r_n + s_n), x_n) \to (B_M(x_{n+1}, r_n + s_n), x_{n+1}) \subset (B_M(x_{n+1}, r_{n+1} + s_{n+1}), x_{n+1}).$$

As M satisfies the local isomorphism property, N is locally isomorphic to M. For $n \in \mathbb{N}$ and $y \neq z$ in $B_N(x, r_n)$, we have $(B_N(y, s_n), y) \ncong (B_N(z, s_n), z)$ since $B_N(y, s_n)$ and $B_N(z, s_n)$ are contained in $B_N(x, r_n + s_n)$ and $(B_N(x, r_n + s_n), x)$ is isomorphic to $(B_M(x_n, r_n + s_n), x_n)$.

For each nontrivial automorphism θ of N, there exist $n \in \mathbb{N}$ and $y \neq z$ in $B_N(x, r_n)$ such that $y\theta = z$. We have $(B_N(y, s_n), y) \cong (B_N(z, s_n), z)$, whence a contradiction. ■

From now on, we consider the metric space (E, δ), the group G and the set Δ defined in Section 1. As a consequence of Theorem 4.1, we have:

Corollary 4.2. Let $\mathcal{T}$ be a Δ-tiling which satisfies the local isomorphism property. Then $\mathcal{T}$ is locally isomorphic to a rigid Δ-tiling if and only if, for each $r \in \mathbb{N}^*$, there exists $T \in \mathcal{T}$ such that no $\sigma \in G - \{\mathrm{Id}\}$ with $\mathcal{T}\sigma = \mathcal{T}$ satisfies $T\sigma \in \mathcal{B}_r^{\mathcal{T}}(T)$.

Remark. If (E, δ) is weakly homogeneous, then Corollary 4.2 and Proposition 1.2 imply that $\mathcal{T}$ is locally isomorphic to a rigid Δ-tiling if and only if, for each $\alpha \in \mathbb{R}_{>0}$, there exists $x \in E$ such that no $\sigma \in G - \{\mathrm{Id}\}$ with $\mathcal{T}\sigma = \mathcal{T}$ satisfies $\delta(x, x\sigma) \leq \alpha$.

Proof of Corollary 4.2. It follows from Theorem 1.3, Corollary 2.2 and Proposition 2.3 that $\mathcal{T}$ is locally isomorphic to a rigid Δ-tiling if and only if the associated $\mathcal{L}_\Delta$-structure is locally isomorphic to a connected rigid $\mathcal{L}_\Delta$-structure, and therefore, by Theorem 4.1, if and only if, for each $r \in \mathbb{N}^*$, there exists $T \in \mathcal{T}$ such that $(\mathcal{T}, U) \cong (\mathcal{T}, V)$ implies $U = V$ for any $U, V \in \mathcal{B}_r^{\mathcal{T}}(T)$.

Now we show that this property is true if and only if, for each $r \in \mathbb{N}^*$, there exists $T \in \mathcal{T}$ such that no $\sigma \in G - \{\mathrm{Id}\}$ with $\mathcal{T}\sigma = \mathcal{T}$ satisfies $T\sigma \in \mathcal{B}_r^{\mathcal{T}}(T)$. For each $T \in \mathcal{T}$, each $r \in \mathbb{N}^*$ and each $\sigma \in G - \{\mathrm{Id}\}$ such that $\mathcal{T}\sigma = \mathcal{T}$, if there exist $U \neq V$ in $\mathcal{B}_r^{\mathcal{T}}(T)$ such that $U\sigma = V$, then we have $T\sigma \in \mathcal{B}_{3r}^{\mathcal{T}}(T)$. Conversely, if $r \geq 2q$ for the integer q of Section 1 and if $T\sigma \in \mathcal{B}_r^{\mathcal{T}}(T)$, then we obtain $U \neq V$ in $\mathcal{B}_r^{\mathcal{T}}(T)$ such that $U\sigma = V$ as follows: we write $U = T$ and $V = T\sigma$ if $T\sigma \neq T$; otherwise, we consider any $U \in \mathcal{B}_q^{\mathcal{T}}(T)$ such that $U\sigma \neq U$, and we write $V = U\sigma$. ■

Corollary 4.3. Let $n \geq 1$ be an integer and let $\mathcal{T}$ be a tiling of the euclidean space $\mathbb{R}^n$ which satisfies the local isomorphism property. Then $\mathcal{T}$ is locally isomorphic to a rigid tiling if and only if it is not invariant through a nontrivial translation.

Proof. By the remark after Corollary 4.2, it suffices to show that, if $\mathcal{T}$ is not invariant through a nontrivial translation, then, for each $\alpha \in \mathbb{R}_{>0}$, there exists $x \in \mathbb{R}^n$ such that $\|xh - x\| > \alpha$ for each nontrivial isometry h of $\mathbb{R}^n$ which stabilizes $\mathcal{T}$. This result follows from Theorem 4.4 below since the isometries of $\mathbb{R}^n$ which stabilize $\mathcal{T}$ form a discrete group by Proposition 1.8. ■

Theorem 4.4. Let $n \geq 1$ be an integer and let H be a discrete group of isometries of the euclidean space $\mathbb{R}^n$. Then H contains no translation if and

only if, for each $\alpha \in \mathbb{R}_{>0}$, there exists $x \in \mathbb{R}^n$ such that $\|xh - x\| > \alpha$ for each $h \in H - \{\mathrm{Id}\}$.

Proof. The "if" part is clear. For the "only if" part, we proceed by induction on n. We denote by E_n the group of all isometries of $\mathbb{R}^n$. For any $X, Y \subset \mathbb{R}^n$, we write $\delta(X,Y) = \inf_{x \in X, y \in Y} \|y - x\|$.

For each $f \in \mathrm{E}_n$, we consider the linear map $\overline{f}$ associated to f, the largest affine subspace W_f of $\mathbb{R}^n$ with $W_f f = W_f$ such that f acts on W_f as a translation, and the element $w_f \in \mathbb{R}^n$ such that $xf = x + w_f$ for each $x \in W_f$. We have $W_f = \{x \in \mathbb{R}^n \mid \|xf - x\| \text{ minimal}\}$.

If V_f is the vector subspace of $\mathbb{R}^n$ orthogonal to W_f and maximal for that property, then the restriction of $\overline{f}$ to V_f is an orthogonal transformation without nontrivial fixed point. Consequently, for each $\alpha \in \mathbb{R}_{>0}$, there exists $\beta \in \mathbb{R}_{>0}$ such that, for each $x \in \mathbb{R}^n$, $\delta(x, W_f) > \beta$ implies $\|xf - x\| > \alpha$.

It follows from a theorem of Bieberbach (see [9, Theorem 1, p. 15]) that H has a normal subgroup N with N abelian and H/N finite. As N is finitely generated, the same properties remain true if we replace N by $N^r = \{h^r \mid h \in N\}$ for an integer $r \geq 2$. Consequently, we can suppose for the remainder of the proof that N is torsion-free. Then we have $w_f \neq 0$ for each $f \in N - \{\mathrm{Id}\}$ since N is discrete like H.

We observe that $W_f g = W_f$ for any $f, g \in \mathrm{E}_n$ which commute, and in particular for $f, g \in N$: We have $(W_f g)f = (W_f f)g = W_f g$ and
$(zg)f - (yg)f = (zf)g - (yf)g = (zf - yf)\overline{g} = (z - y)\overline{g} = zg - yg$
for any $y, z \in W_f$. It follows that f stabilizes $W_f g$ and acts on $W_f g$ as a translation, which implies $W_f g = W_f$.

We consider $W = \cap_{f \in N} W_f$. In order to prove that W is nonempty, it suffices to show that, for each $r \in \mathbb{N}^*$ and any $f_1, ..., f_{r+1} \in N$, if $W_r = W_{f_1} \cap ... \cap W_{f_r}$ is nonempty, then $W_{r+1} = W_{f_1} \cap ... \cap W_{f_{r+1}}$ is nonempty. But we have $W_r f_{r+1} = W_r$ since $W_{f_i} f_{r+1} = W_{f_i}$ for $1 \leq i \leq r$; it follows that W_{r+1} is the largest affine subspace V of W_r with $V f_{r+1} = V$ such that f_{r+1} acts on V as a translation.

Now we show that $Wh = W$ for each $h \in H$. For each $f \in N$, we have $W(hfh^{-1}) = W$ since $hfh^{-1} \in N$, and therefore $Whf = W(hfh^{-1})h = Wh$. Moreover, for each $f \in N$ and any $y, z \in W$, we have
$zhf - yhf = z(hfh^{-1})h - y(hfh^{-1})h = [z(hfh^{-1}) - y(hfh^{-1})]\overline{h}$
$= (z - y)\overline{h} = zh - yh$
since $hfh^{-1} \in N$. It follows that each $f \in N$ stabilizes Wh and acts on Wh as a translation, which implies $Wh = W$.

Now we fix $\alpha \in \mathbb{R}_{>0}$ and we prove that there exists $x \in \mathbb{R}^n$ such that $\|xh - x\| > \alpha$ for each $h \in H - \{\mathrm{Id}\}$. We consider the set Ω of all affine subspaces of $\mathbb{R}^n$ which are orthogonal to W and maximal for that property. For each $U \in \Omega$ and each $h \in H$, we have $Uh \in \Omega$ since $Wh = W$.

First we show that $\{g \in H \mid \delta(U, Ug) \leq \alpha\}$ is finite for each $U \in \Omega$. As each $h \in N$ is acting on W as a translation of vector w_h, we have $\delta(V, Vh) = \|w_h\|$ for $V \in \Omega$ and $h \in N$, and $w_{gh} = w_g + w_h$ for $g, h \in N$. We write $\gamma = \inf_{h \in N - \{\mathrm{Id}\}} \|w_h\|$. We have $\gamma \neq 0$ since N is discrete and torsion-free. We consider $r \in \mathbb{N}^*$ such that $r\gamma > 2\alpha$. For each $U \in \Omega$, each $g \in H$ such that $\delta(U, Ug) \leq \alpha$ and each $h \in N - \{\mathrm{Id}\}$, the inequality $\delta(Ug, Ugh^r) = \|w_{h^r}\| = r\,\|w_h\| \geq r\gamma > 2\alpha$ implies $\delta(U, Ugh^r) > \alpha$. As N^r has finite index in N and therefore in H, it follows that $\{g \in H \mid \delta(U, Ug) \leq \alpha\}$ is finite.

Now we consider $K = \{h \in H \mid xh - x \in W \text{ for each } x \in \mathbb{R}^n\}$ and, for each $h \in K$, the restriction h_W of h to W. For each $x \in \mathbb{R}^n$, we denote by x_W the projection of x on W. Then $K_W = \{h_W \mid h \in K\}$ is, like K, a discrete group of isometries without nontrivial translation, since $xh - x = x_W h - x_W$ for each $h \in K$ and each $x \in \mathbb{R}^n$. The induction hypothesis applied to W and K_W implies that there exists $x \in W$ such that $\|xh_W - x\| > \alpha$ for each $h \in K - \{\mathrm{Id}\}$.

We consider the unique $U \in \Omega$ such that $x \in U$. We have $\delta(U, Uh) = \|xh_W - x\| > \alpha$ for each $h \in K - \{\mathrm{Id}\}$. For each $h \in H - K$, we have $U \cap W_h \subsetneq U$ and $S_h = \{y \in U \mid \|yh - y\| \leq \alpha\}$ is contained in $A + (U \cap W_h)$ for a bounded subset A of U since there exists $\beta \in \mathbb{R}_{>0}$ such that, for each $x \in \mathbb{R}^n$, $\delta(x, W_h) > \beta$ implies $\|xh - x\| > \alpha$. As S_h is empty for each $h \in H$ such that $\delta(U, Uh) > \alpha$, there exist finitely many nonempty subsets S_h, and their union cannot cover U. ∎

Now, we illustrate Corollary 4.2 and Corollary 4.3 with three examples related to aperiodicity. Several different definitions have been given for that notion (see [4, p. 4]).

We consider the system Δ defined in Section 1 and the set $\mathcal{C}$ of all Δ-tilings which satisfy a set Ω of *local rules*, each of them saying which configurations of some given size can appear in a tiling belonging to $\mathcal{C}$. According to [12, p. 208], we say that Ω is *strong* if the Δ-tilings in $\mathcal{C}$ satisfy the local isomorphism property and if they are not invariant through any nontrivial translation. We use the classical definition of translation for the euclidean spaces $\mathbb{R}^n$, and the definition given in Section 3 for the general case.

Here we do not suppose Ω finite. One reason is that some natural sets of tilings are defined by strong infinite sets of local rules (for instance we showed this property in [8] for the set of all complete folding sequences, and for the set of all coverings of the plane by complete folding curves which satisfy the local isomorphism property). Another reason is that, for each Δ-tiling $\mathcal{T}$ which satisfies the local isomorphism property, the set of all Δ-tilings which are locally isomorphic to $\mathcal{T}$ is defined by a set of local rules which can be finite as in Examples 2 and 3, or infinite as in Example 1.

By Corollary 4.3, any tiling of an euclidean space of finite dimension which

satisfies a strong set of local rules is locally isomorphic to a rigid tiling if and only if it is not invariant through a nontrivial translation. We do not know presently if this result can be generalized with the notion of translation that we consider.

In Examples 1 and 3 below, the group G consists of all isometries of E; in Example 2 we only consider positive isometries of $\mathbb{R}^3$. In each example, all the structures are uniformly locally finite and satisfy the local isomorphism property. On the other hand, they do not satisfy (P). Some of them are rigid and others have nontrivial automorphisms, but all of them are locally isomorphic.

Example 1. For each $r \in \mathbb{R} - \mathbb{Q}$ and each $s \in \mathbb{R}$, we consider the line $L(r,s)$ of equation $y = rx + s$. We write
$\Omega(r,s) = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} \mid L(r,s)\cap([a-1/2, a+1/2[\times[b-1/2, b+1/2[) \neq \varnothing\}$.
For each $a \in \mathbb{Z}$, $\{a\} \times \mathbb{Z}$ contains $n+1$ or $n+2$ points of $\Omega(r,s)$, where n is the integral part of $|r|$. We colour the point a in white if $\{a\} \times \mathbb{Z}$ contains $n+1$ points of $\Omega(r,s)$ and in black otherwise. We consider the tiling $\mathcal{T}(r,s)$ of the euclidean space $\mathbb{R}$ which consists of the segments $[a, a+1]$ for $a \in \mathbb{Z}$ with their endpoints coloured in white or black as above.

For each $r \in \mathbb{R} - \mathbb{Q}$, the tilings $\mathcal{T}(r,s)$ are locally isomorphic and each of them satisfies the local isomorphism property. They do not satisfy (P) since they are not invariant through any nontrivial translation. They are invariant through a unique symmetry if there exist $a, b \in \mathbb{Z}$ such that (a,b) or $(a+1/2, b)$ or $(a, b+1/2)$ belongs to $L(r,s)$, and rigid otherwise. The three possibilities above are respectively realized for $s \in \mathbb{Z} + r\mathbb{Z}$, $s \in r/2 + \mathbb{Z} + r\mathbb{Z}$, $s \in 1/2 + \mathbb{Z} + r\mathbb{Z}$.

Example 2. Let (E,δ) be the euclidean space $\mathbb{R}^3$ and let G consist of the positive isometries. Let T be a tiling of $\mathbb{R}^3$ which satisfies the local isomorphism property. Suppose that the group of isometries which leave $\mathcal{T}$ globally invariant is generated by a "screwing motion" σ, which is the composition of a translation with a rotation about an axis parallel to the translation. If the angle of the rotation belongs to $\pi\mathbb{Q}$, then some nontrivial power of σ is a translation, and $\mathcal{T}$ satisfies (P). Otherwise, $\mathcal{T}$ does not satisfy (P), and Theorem 4.1 implies that $\mathcal{T}$ is locally isomorphic to a rigid tiling. According to [12, Section 7.2, pp. 208-213], examples of that situation have been given by Danzer for tilings obtained from 1 prototile (the examples with n odd must be considered).

Example 3. In 1979, R. Penrose gave his famous example (see [3]) of two polygonal prototiles, the "arrow" and the "kite", which define an aperiodic class of tilings of the euclidean space $\mathbb{R}^2$. There exist 2^ω Penrose tilings. All of them are locally isomorphic and each of them satisfies the local isomorphism

property. The Robinson tilings (see [10]) have the same properties, but they are constructed from a larger set of prototiles. Penrose asked if there exist classes of tilings of $\mathbb{R}^2$ defined from a single prototile which have these properties. The question is apparently still open for tilings with non-overlapping tiles (see [6]).

In the hyperbolic plane, it is not difficult to construct such an example. Here, we use the representation of the hyperbolic plane by the Poincaré half-plane $\mathbb{R} \times \mathbb{R}_{>0}$.

Figure 1 illustrates the construction of the tilings in our example, which is a particular case of those given in [6]. We denote by Ω the set of all tilings constructed in that way. For any $\mathcal{S}, \mathcal{T} \in \Omega$, each $S \in \mathcal{S}$ and each $T \in \mathcal{T}$, there exists a unique $\sigma \in G$ such that $S\sigma = T$, because of the arrows on the edges of the tiles.

For each $\mathcal{T} \in \Omega$ and each $T \in \mathcal{T}$, we consider $(U_n(T))_{n \in \mathbb{N}}$ where $U_0(T) = T$ and, for each $n \in \mathbb{N}$, $U_{n+1}(T)$ is the tile just above $U_n(T)$. For each $n \in \mathbb{N}^*$, we write $a_n(T) = 0$ if $U_n(T)$ is at the left of $U_{n-1}(T)$, and $a_n(T) = 1$ otherwise.

For any $\mathcal{S}, \mathcal{T} \in \Omega$, each $S \in \mathcal{S}$ and each $T \in \mathcal{T}$, we have $(\mathcal{S}, S) \cong (\mathcal{T}, T)$ if and only if $(a_n(S))_{n \in \mathbb{N}^*} = (a_n(T))_{n \in \mathbb{N}^*}$. For each $r \in \mathbb{N}^*$, we have $(\mathcal{B}_r^{\mathcal{S}}(S), S) \cong (\mathcal{B}_r^{\mathcal{T}}(T), T)$ if and only if $(a_1(S), ..., a_r(S)) = (a_1(T), ..., a_r(T))$.

For any $\mathcal{S}, \mathcal{T} \in \Omega$, each $S \in \mathcal{S}$, each $T \in \mathcal{T}$ and each $r \in \mathbb{N}^*$, there exists $S' \in \mathcal{T}$ such that $T = U_r(S')$ and $(a_1(S'), ..., a_r(S')) = (a_1(S), ..., a_r(S))$, which implies $S' \in \mathcal{B}_r^{\mathcal{T}}(T)$ and $(\mathcal{B}_r^{\mathcal{S}}(S), S) \cong (\mathcal{B}_r^{\mathcal{T}}(S'), S')$. Consequently, any tiling in Ω satisfies the local isomorphism property, and any two such tilings are locally isomorphic.

For each $\mathcal{T} \in \Omega$ and any $S, T \in \mathcal{T}$, there exist $i, j \in \mathbb{N}$ such that $U_i(S) = U_j(T)$; for $n \geq i+1$, we have $a_n(S) = a_{n+k}(T)$ where $k = j - i$. Consequently, each $\mathcal{T} \in \Omega$ only realizes countably many sequences $(a_n)_{n \in \mathbb{N}^*} \in \{0,1\}^{\mathbb{N}^*}$. As each such sequence is realized by a tiling $\mathcal{T} \in \Omega$, it follows that Ω is the union of 2^ω isomorphism classes. This property is a particular case of Corollary 2.6 above.

Now we show that, for each $\mathcal{T} \in \Omega$ which is not rigid and each $T \in \mathcal{T}$, there exists $k \in \mathbb{N}^*$ such that $a_n(T) = a_{n+k}(T)$ for n large enough: We consider $\sigma \in G - \{\mathrm{Id}\}$ such that $\mathcal{T}\sigma = \mathcal{T}$ and $i, j \in \mathbb{N}$ such that $U_i(T) = U_j(T\sigma)$. We have $i \neq j$ because $U_i(T) = U_i(T\sigma) = U_i(T)\sigma$ would imply $\sigma = \mathrm{Id}$. For $k = j - i$, we have $U_n(T) = U_{n+k}(T\sigma)$ for $n \geq i$, and therefore $a_n(T) = a_{n+k}(T\sigma) = a_{n+k}(T)$ for $n \geq i + 1$.

Conversely, for each $\mathcal{T} \in \Omega$ and each $T \in \mathcal{T}$, if $I = \{k \in \mathbb{Z} \mid a_n(T) = a_{n+k}(T)$ for n large enough$\}$ contains a nonzero element, then I is the ideal of $\mathbb{Z}$ generated by the smallest $h \in \mathbb{N}^*$ which belongs to I. For each $r \in \mathbb{N}$ such that $a_n(T) = a_{n+h}(T)$ for $n > r$, the isometry which sends $U_r(T)$ to $U_{r+h}(T)$ generates $\{\sigma \in G \mid \mathcal{T}\sigma = \mathcal{T}\}$.

Remark. In Example 3, similar to the case of Penrose tilings or Robinson tilings, the class of Δ-tilings that we consider is defined by a local rule which describes the possible configurations of the immediate neighbours of a tile. In the case of Penrose tilings or Robinson tilings (see [7, p. 125]), it follows that there exists a local rule expressed by one sentence which characterizes among the connected $\mathcal{L}_\Delta$-structures those which are associated to Δ-tilings, because no such tiling is invariant through an infinite group of isometries. On the other hand, in Example 3, no such rule exists since any local rule satisfied by the $\mathcal{L}_\Delta$-structures associated to Δ-tilings is also satisfied by some of their quotients.

The following example generalizes the argument of Example 3, even though the relational structures that we consider are not represented by tilings:

Example 4. We write $\mathcal{L} = \{P_1, ..., P_k\}$ where $P_1, ..., P_k$ are unary functional symbols, and we consider the nonempty $\mathcal{L}$-structures M which satisfy the following properties:
1) For each $x \in M$, there exists one and only one pair (i, y) with $1 \leq i \leq k$ and $y \in M$ such that $yP_i = x$;
2) $xP_{i_1}...P_{i_r} = x$ implies $r = 0$ for $r \in \mathbb{N}$, $1 \leq i_1, ..., i_r \leq k$ and $x \in M$.
Each connected such structure induces a directed tree where the pairs (x, y) of consecutive vertices are characterized by the existence of a unique $i \in \{1, ..., k\}$ such that $xP_i = y$; each vertex is the origin of k edges.

In order to apply the results of the present paper, it is convenient to consider $P_1, ..., P_k$ as binary relations. Similarly to Example 3, the nonempty $\mathcal{L}$-structures which satisfy 1) and 2) are locally isomorphic, and each of them satisfies the local isomorphism property. In fact, for any such structures M, N, each $x \in M$, each $y \in N$ and each $r \in \mathbb{N}^*$, we have $(B_M(x, r), x) \cong (B_M(z, r), z)$ for $z = yP_{i_r}...P_{i_1}$ where $i_1, ..., i_r$ are the elements of $\{1, ..., k\}$ such that $xP_{i_1}^{-1}...P_{i_r}^{-1}$ exists.

Now, for each nonempty connected $\mathcal{L}$-structure M which satisfies 1), 2) and each $x \in M$, we consider the sequence $(i_r(x))_{r\in\mathbb{N}} \in \{1, ..., k\}^{\mathbb{N}}$ such that $xP_{i_1(x)}^{-1}...P_{i_r(x)}^{-1}$ exists for each $r \in \mathbb{N}$. Similarly to Example 3, M has a nontrivial automorphism if and only if there exists an integer k such that $i_r(x) = i_{r+k}(x)$ for r large enough. In that case, the group of automorphisms of M is infinite cyclic.

By Theorem 4.1, it follows that the nonempty $\mathcal{L}$-structures which satisfy 1) and 2) are rigid.

The last two examples are not related to tilings. They are given in order to illustrate the importance of each hypothesis in Theorem 4.1.

Example 5. The *Cayley graph* of a group G relative to a generating family $(x_i)_{i\in I}$ is the relational structure M defined on G as follows: for $i \in I$ and

$y, z \in G$, we write $R_i(y,z)$ if and only if $z = yx_i$. The structure M is uniformly locally finite if I is finite. The automorphisms of M are the maps $y \to gy$ for $g \in G$. For any $y, z \in M$, we have $(M, y) \cong (M, z)$ since there exists $g \in G$ such that $gy = z$. In particular, M satisfies the local isomorphism property and M is not locally isomorphic to a rigid structure. If G is freely generated by the elements x_i, then M is not invariant through any nontrivial translation since, for each $g \in G$ and each $r \in \mathbb{N}$, there exists $x \in M$ such that $gx \notin B_M(x, r)$.

Example 6. Here, the language $\mathcal{L}$ consists of one binary relational symbol. The $\mathcal{L}$-structure M shown by Figure 2 is uniformly locally finite, but it does not satisfy the local isomorphism property. The only automorphisms of M are the maps $x_{i,j} \to x_{i+k,j}$ for $k \in \mathbb{Z}$. Consequently, M does not satisfy the characterization of Theorem 4.1. Anyway, each connected $\mathcal{L}$-structure N locally isomorphic to M is isomorphic to M, and therefore not rigid.

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